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ON THE GEOMETRIC MEAN OF CHARACTER DEGREES OF A FINITE GROUP

NGUYEN NGOC HUNG

Department of Mathematics,
The University of Akron,
Akron, Ohio 44325, USA

Abstract. Over the past decade, several interesting results on the so-called average character degree of a finite group and its connections with the group structure have been found. In this paper we introduce the geometric mean of character degrees and prove that the alternating group A_5, which is the smallest non-solvable group, has minimal geometric mean of irreducible character degrees among all non-solvable groups.

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1. Introduction

The study of character degrees and character tables of finite groups and their connections with group structure is an active line of research in the area of representation theory of finite groups. Over the past decade, several interesting results on the so-called average character degree have been found. In particular, it was shown in [10, 7, 12] that when a finite group has a relatively small average degree of irreducible characters, the group must be solvable. These results are motivated by the fact that when the average character degree of a finite group is close to 1, the group is somewhat close to abelian.

In this paper we initiate the study of another kind of average of character degrees, namely the geometric mean. For a finite group G, let $\mathrm{Irr}(G)$ denote the set of all ordinary irreducible characters of G, and set

$$\mathrm{gm}(G) := \left(\prod_{\chi \in \mathrm{Irr}(G)} \chi(1) \right)^{1/|\mathrm{Irr}(G)|}.$$

E-mail address: `hungnguyen@uakron.edu`.

Theorem 1. *Let G be a finite group. If $\mathrm{gm}(G) < 180^{1/5}$, then G is solvable.*

Note that $180^{1/5}$ is exactly the geometric mean of character degrees of A_5 – the alternating group of degree 5. Theorem 1 basically says that A_5, which happens to be the smallest non-solvable group, has the minimal geometric mean character degree among all non-solvable groups. It seems to us that this phenomenon still holds true when solvability is replaced by other characteristics of finite groups like nilpotency, supersolvability, having normal Sylow p-subgroups, or having normal p-complements, see Section 3 for more detailed discussion.

2. Solvability and geometric mean of character degrees

We start with a result on extendibility of characters of finite simple groups. Naturally, its proof relies on the classification of finite simple groups.

Lemma 2. *Suppose that $S \neq \mathsf{A}_5$ is a non-abelian finite simple group. Then there exists $\lambda \in \mathrm{Irr}(S)$ such that $\lambda(1) \geq 8$ and λ is extendible to the automorphism group $\mathrm{Aut}(S)$.*

Proof. It is routine to check this for $\mathrm{PSL}(2,7) \cong \mathrm{PSL}(3,2)$, A_7, simple sporadic groups, and the Tits groups by using [2]. For the alternating groups A_n with $n \geq 9$, we choose λ to be the character of degree $n-1$ corresponding to the partition $(n-1,1)$. Finally, for simple groups of Lie type (different from $\mathrm{PSL}(2,7)$) in characteristic p, the required character λ can be chosen to be the Steinberg character of S of degree $|S|_p \geq 8$. (see [3] for the fact that the Steinberg character of S is extendible to $\mathrm{Aut}(S)$.) $\square$

Lemma 3. *Suppose that $N \neq \mathsf{A}_5$ is a non-abelian minimal normal subgroup of a finite group G. Then there exists $\theta \in \mathrm{Irr}(N)$ such that $\theta(1) \geq 8$ and θ is extendible to G.*

Proof. Suppose that N is a direct product of k copies of a non-abelian simple group S, where $(S,k) \neq (\mathsf{A}_5, 1)$. First we consider the case $S = \mathsf{A}_5$ and $k \geq 2$. Let λ be the unique irreducible character of S of degree 5 and let $\theta \in \mathrm{Irr}(N)$ be the tensor product of k copies of λ. We then have $\theta(1) = 5^k \geq 25$. Furthermore, since λ is extendible to $\mathrm{Aut}(S)$, the character θ extends to G and we are done in this case, see [1, Lemma 5] for instance.

For the remaining of the proof we may assume that $S \neq \mathsf{A}_5$. Let $\lambda \in \mathrm{Irr}(S)$ be a character produced by Lemma 2. Now we take $\theta \in \mathrm{Irr}(N)$ to be the tensor product of k copies of λ to have that θ extends to G. $\square$

For d a positive integer, we denote by $n_d(G)$ the number of irreducible characters of G of degree d. For $N \trianglelefteq G$, we write $\mathrm{Irr}(G|N) := \{\chi \in \mathrm{Irr}(G) \mid N \nsubseteq \mathrm{Ker}(\chi)\}$ and $n_d(G|N) := |\{\chi \in \mathrm{Irr}(G|N)) \mid \chi(1) = d\}|$. We also write $\mathrm{gm}(G|N)$ to denote the geometric mean of the degrees of the characters in $\mathrm{Irr}(G|N)$.

Lemma 4. *Let G be a finite group such that $\mathsf{A}_5 \subseteq [G,G]$-the commutator subgroup of G and $\mathsf{A}_5 \trianglelefteq G$. Then $\mathrm{gm}(G) \geq \mathrm{gm}(\mathsf{A}_5)$.*

Proof. Let $C := \mathbf{C}_G(\mathsf{A}_5)$. Then G/C is an almost simple group with socle A_5. If $G/C \cong \mathsf{A}_5$ then we have $G \cong \mathsf{A}_5 \times C$ and the lemma follows easily.

It remains to assume that $G/C \cong \mathsf{S}_5$. In particular, $\mathsf{A}_5 \times C$ can be viewed as a subgroup of index 2 in G. As $n_1(\mathsf{A}_5 \times C) = n_1(C)$, it follows that $n_1(G) \leq 2n_1(C)$.

On the other hand, as $n_3(\mathsf{A}_5) = 2$, we have $n_3(\mathsf{A}_5 \times C) \geq 2n_1(C)$. We deduce that

$$n_3(\mathsf{A}_5 \times C) \geq n_1(G).$$

Note that if χ is an irreducible character of G then $\chi_{\mathsf{A}_5 \times C}$ is either irreducible or a sum of two distinct irreducible characters of $\mathsf{A}_5 \times C$. Therefore the above inequality implies that

$$\frac{n_3(G)}{2} + 2n_6(G) \geq n_1(G).$$

We observe that A_5 has two irreducible characters of degrees 4 and 5, each of which is extendible to S_5. We therefore obtain injections from linear characters of G/A_5 to the set of irreducible characters of degree 4 as well as the set of irreducible characters of degree 5 of G. It follows that

$$n_4(G) \geq n_1(G/\mathsf{A}_5) \text{ and } n_5(G) \geq n_1(G/\mathsf{A}_5).$$

Since $\mathsf{A}_5 \subseteq G'$, we see that every linear character of G has A_5 inside its kernel so that $n_1(G/\mathsf{A}_5) = n_1(G)$. Therefore

$$n_4(G) \geq n_1(G) \text{ and } n_5(G) \geq n_1(G).$$

We now have

$$\begin{aligned}
&(3^{n_3(G)} \cdot 4^{n_4(G)} \cdot 5^{n_5(G)} \cdot 6^{n_6(G)})^{\frac{1}{n_1(G)+n_3(G)+n_4(G)+n_5(G)+n_6(G)}} \\
&\geq (3^{n_3(G)} \cdot 4^{n_1(G)} \cdot 5^{n_1(G)} \cdot 6^{n_6(G)})^{\frac{1}{3n_1(G)+n_3(G)+n_6(G)}} \\
&\geq (3^{2n_1(G)} \cdot 4^{n_1(G)} \cdot 5^{n_1(G)})^{\frac{1}{5n_1(G)}} \\
&= (3^2 \cdot 4 \cdot 5)^{\frac{1}{5}} = \mathrm{gm}(\mathsf{A}_5),
\end{aligned}$$

which means that the geometric mean degree of irreducible characters of degrees 1, 3, 4, 5, and 6 of G is at least $\mathrm{gm}(\mathsf{A}_5)$.

We also observe that A_5 does not have an irreducible character of degree 2 and hence A_5 is contained in the kernel of every irreducible character of degree 2 of G. Equivalently, we have $n_2(G/N) = n_2(G)$. Using the fact that A_5 has an irreducible character of degree 5 which is extendible to S_5, we get

$$n_{10}(G) \geq n_2(G/N) = n_2(G).$$

Therefore, the geometric mean of the degrees of irreducible characters of degrees 2 and 10 of G is at least $\sqrt{20}$, which is greater than $\mathrm{gm}(\mathsf{A}_5)$.

The lemma follows from the conclusions of the last two paragraphs. □

We now can prove a major case of the main result.

Lemma 5. *If G has a non-abelian minimal normal subgroup N such that $N \subseteq G'$, then* $\mathrm{gm}(G) \geq \mathrm{gm}(\mathsf{A}_5)$.

Proof. Since $N \subseteq G'$, we see that every linear character of G has its kernel containing N so that $n_1(G/N) = n_1(G)$. Moreover, as it is well-known that every simple group does not have an irreducible character of degree 2 (see [6, Problem 3.3]), the group N does not have an irreducible character of degree 2 neither. This implies that N is contained in the kernel of every irreducible character of degree 2 of G. Equivalently, we have $n_2(G/N) = n_2(G)$.

As the case $N = \mathsf{A}_5$ was already considered in Lemma 4, we can assume from now on that $N \neq \mathsf{A}_5$. Then Lemma 3 implies that there exists $\theta \in \mathrm{Irr}(N)$ such that $\theta(1) \geq 8$ and θ is extended to $\chi \in \mathrm{Irr}(G)$. By Gallagher's lemma (see [6, Corollary

6.17]), we have a bijection $\beta \mapsto \chi\beta$ from $\mathrm{Irr}(G/N)$ to the set of irreducible characters of G lying over θ. In particular, we have

$$n_D(G) \geq n_1(G/N) = n_1(G)$$

and

$$n_{2D}(G) \geq n_2(G/N) = n_2(G),$$

where $D := \theta(1) \geq 8$. It follows that the geometric mean of the degrees of all irreducible characters of G of degrees 1 and D is at least $\sqrt{D}$, which is greater than $\mathrm{gm}(\mathsf{A}_5)$. Also, the geometric mean of the degrees of all irreducible characters of G of degrees 2 and $2D$ is at least $\sqrt{4D}$, which is greater than $\mathrm{gm}(\mathsf{A}_5)$ as well. Remark that $\mathrm{gm}(\mathsf{A}_5) < 3$. We conclude that $\mathrm{gm}(G) \geq \mathrm{gm}(\mathsf{A}_5)$, as desired. □

Lemma 6. *Suppose that G is an internal central product of $M \cong \mathrm{SL}(2,5)$ and a finite group C with the central amalgamated subgroup $\mathbf{Z}(M) = M \cap C$. Suppose furthermore that $M \subseteq G'$ and that there is $\chi \in \mathrm{Irr}(G)$ of order 2 with $\mathbf{Z}(M) \nsubseteq \mathrm{Ker}(\chi)$. Then $\mathrm{gm}(G) > \mathrm{gm}(\mathsf{A}_5)$.*

Proof. For simplicity, put $Z := \mathbf{Z}(M)$. Let a be the only nontrivial element of Z. The irreducible characters of G are in one-to-one correspondence with those $(\alpha, \beta) \in \mathrm{Irr}(M) \times \mathrm{Irr}(C)$ such that $\alpha(a) = \beta(a) \in \{1, -1\}$. Furthermore, if $\psi \in \mathrm{Irr}(G)$ corresponds to (α, β) then $\psi(1) = \alpha(1)\beta(1)$.

Recall that $\chi(1) = 2$ and $\chi \in \mathrm{Irr}(G|Z)$. Therefore, if $(\alpha, \beta) \in \mathrm{Irr}(M) \times \mathrm{Irr}(C)$ corresponds to χ, then Z is not contained in the kernel of $\alpha \in \mathrm{Irr}(M) = \mathrm{Irr}(\mathrm{SL}(2,5))$ so that $\alpha(1) \geq 2$. Therefore $\beta(1) = 1$ which implies that $\beta \in \mathrm{Irr}(C|Z)$ is an extension of the unique non-principal linear character of Z. By Gallagher's lemma, we obtain a degree-preserving bijection from $\mathrm{Irr}(C/Z)$ to $\mathrm{Irr}(C|Z)$. Using the facts $C/Z \cong G/M$ and $M \subseteq G'$, we deduce that

$$n_1(C|Z) = n_1(C/Z) = n_1(G/M) = n_1(G).$$

Claim 1: The geometric mean of the degrees of irreducible characters of G of degrees 1, 4, and 5 is greater than $\mathrm{gm}(\mathsf{A}_5)$.

Proof: Since $n_4(M/Z) = n_4(M|Z) = 1$, we have

$$n_4(G|Z) \geq n_4(M|Z)n_1(C|Z) = n_1(C|Z) = n_1(G)$$

and

$$n_4(G/Z) \geq n_4(M/Z)n_1(C/Z) = n_1(C/Z) = n_1(G).$$

Therefore

$$n_4(G) = n_4(G|Z) + n_4(G/Z) \geq 2n_1(G).$$

Similarly, $n_5(G)$ can be estimated as follows

$$n_5(G) \geq n_5(G/Z) \geq n_5(M/Z)n_1(C/Z) = n_1(C/Z) = n_1(G).$$

We conclude that the geometric mean of the degrees of irreducible characters of G os degrees 1, 4, and 5 is at least $\sqrt[4]{1 \cdot 4^2 \cdot 5}$, which is greater than $\mathrm{gm}(\mathsf{A}_5)$.

Claim 2: The geometric mean of the degrees of irreducible characters of G of degrees 2 and 6 is greater than $\mathrm{gm}(\mathsf{A}_5)$.

Proof: As $n_1(M|Z) = 0$ and $n_2(M|Z) = 2$, we have

$$n_2(G|Z) = n_1(M|Z)n_2(C|Z) + n_2(M|Z)n_1(C|Z) = 2n_1(C|Z) = 2n_1(G).$$

Also, as $n_1(M/Z) = 1$ and $n_2(M/Z) = 0$, we have

$$n_2(G/Z) = n_1(M/Z)n_2(C/Z) + n_2(M/Z)n_1(C/Z) = n_2(C/Z).$$

It follows that

$$n_2(G) = n_2(G/Z) + n_2(G|Z) = n_2(C/Z) + 2n_1(G),$$

which implies that $n_2(C/Z) = n_2(G) - 2n_1(G)$ and in particular $n_2(G) \geq n_1(G)$.

We now need to estimate $n_6(G)$. We have

$$n_6(G|Z) \geq n_6(M|Z)n_1(C|Z) = n_1(C|Z) = n_1(G)$$

since $n_6(M|Z) = 1$, and

$$n_6(G/Z) \geq n_3(M/Z)n_2(C/Z) = 2n_2(C/Z) = 2(n_2(G) - 2n_1(G)).$$

Therefore

$$\begin{aligned} n_6(G) &= n_6(G|Z) + n_6(G/Z) \\ &\geq n_1(G) + 2(n_2(G) - 2n_1(G)) \\ &= 2n_2(G) - 3n_1(G). \end{aligned}$$

As $n_2(G) \geq 2n_1(G)$, it follows that

$$n_6(G) \geq \frac{1}{2}n_2(G).$$

We conclude that the geometric mean of the degrees of irreducible characters of G of degrees 2 and 6 is at least $\sqrt[3]{2^2 \cdot 6}$, which is greater than $\mathrm{gm}(\mathsf{A}_5)$ as well.

Claims 1 and 2 imply that $\mathrm{gm}(G) > \mathrm{gm}(\mathsf{A}_5)$, as the lemma stated. □

We are now in the position to prove Theorem 1.

Proof of Theorem 1. Assume, to the contrary, that the theorem is false and let G be a minimal counterexample. In particular, we have that G is non-solvable and $\mathrm{gm}(G) < \mathrm{gm}(\mathsf{A}_5)$. Let $M \trianglelefteq G$ be minimal such that M is non-solvable. Then M is perfect and contained in the last term of the derived series of G. Take $N \subseteq M$ to be a minimal normal subgroup of G. Then we have $N \subseteq M = M' \subseteq G'$.

If N is non-abelian then we would have $\mathrm{gm}(G) \geq \mathrm{gm}(\mathsf{A}_5)$ by Lemma 5, which is not the case. Therefore we assume from now on that N is abelian. It follows that G/N is non-solvable by the non-solvability of G. By the minimality of G, we then deduce that $\mathrm{gm}(G/N) \geq \mathrm{gm}(\mathsf{A}_5)$.

Recall that $N \subseteq G'$ and thus $n_1(G/N) = n_1(G)$. We claim that $n_2(G/N) < n_2(G)$. Assume otherwise that $n_2(G/N) = n_2(G)$. Then the degree of every character in $\mathrm{Irr}(G|N)$ is at least 3 and therefore we have

$$\mathrm{gm}(G) \geq \min\{\mathrm{gm}(G/N), \mathrm{gm}(G|N)\} \geq \min\{\mathrm{gm}(\mathsf{A}_5, 3)\} = \mathrm{gm}(\mathsf{A}_5),$$

which violates our assumption. So the claim is proved. In other words, there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = 2$ and $N \nsubseteq \mathrm{Ker}(\chi)$.

Let C be the subgroup of G such that $C/\mathrm{Ker}(\chi) = \mathbf{Z}(G/\mathrm{Ker}(\chi))$. It was shown in the proof of [7, Theorem 2.2] that $M \cong \mathrm{SL}(2,5)$ and $G = MC$ is a central product with the central amalgamated subgroup $\mathbf{Z}(M) = M \cap C$. We now apply Lemma 6 to deduce that $\mathrm{gm}(G) > \mathrm{gm}(\mathsf{A}_5)$ and this contradiction completes the proof. □

3. Concluding remarks

We make some further observation on the values of $\mathrm{gm}(G)$ and its connection with the structure of G.

The set

$$\{\mathrm{gm}(G) : G \text{ is a group of prime power order}\}$$

of geometric means of character degrees of finite groups is dense in the set of real numbers at least 1. To see that, we consider the following example.

For an odd prime p and a positive integer n let $E_{p,n}$ be the direct product of n copies of an extraspecial p-group of exponent p and order p^3. The group $E_{p,1}$ has p^2 linear characters and $p-1$ complex irreducible characters of degree p. Thus $\mathrm{gm}(E_{p,1}) = p^{(p-1)/(p^2+p-1)}$. Clearly $\mathrm{gm}(E_{p,n}) = \mathrm{gm}(E_{p,1})^n$.

Let (K, L) be any interval with $1 \leq K < L$. We claim that there are n and p so that $\mathrm{gm}(E_{p,n})$ is in the interval (K, L). For this it is sufficient to see that $\log(\mathrm{gm}(E_{p,n})) = n \cdot ((p-1)/(p^2+p-1)) \log(p)$ is in $(\log(K), \log(L))$ for some n and p.

Notice that $((p-1)/(p^2+p-1)) \log(p)$ tends to 0 as p goes to infinity. So we may choose a p so that $((p-1)/(p^2+p-1)) \log(p) < \min\{\log(K), \log(L) - \log(K)\}$. But then for this p there must exist a suitable n.

Now let $\mathcal{P}$ be a certain characteristic of a finite group, like having a normal Sylow p-subgroup or having a normal p-complement. Let $G(\mathcal{P})$ be a finite group of smallest order without characteristic $\mathcal{P}$ and $\mathrm{gm}_{\mathcal{P}}(G)$ denote the geometric mean degree of certain irreducible characters of G that are defined in terms of $\mathcal{P}$. As mentioned in the introduction, we believe that

if $\mathrm{gm}_{\mathcal{P}}(G) < \mathrm{gm}(G(\mathcal{P}))$ then G has characteristic $\mathcal{P}$.

For a given prime p, we define $\alpha(p)$ to be the smallest positive integer α such that $\alpha(p)p + 1$ is a prime power. Consequently, there is a prime ℓ and a positive integer k such that $\alpha(p)p = \ell^k - 1$. Note that the cyclic group C_{ℓ^k-1} admits a faithful action on the elementary abelian ℓ-group C_ℓ^k, we can form a semi-direct product

$$H := C_\ell^k \rtimes C_{\ell^k-1}.$$

It is an easy exercise to show that H is the smallest group without a normal Sylow p-subgroup. Also, H has $\alpha(p)p$ linear characters and one nonlinear irreducible character of degree $\alpha(p)p$. Following [5], for a group G and a prime p, we define

$$\mathrm{Irr}_p(G) := \{\chi \in \mathrm{Irr}(G) \mid \chi(1) = 1 \text{ or } p \mid \chi(1)\}$$

and

$$\mathrm{gm}_p(G) := \left(\prod_{\chi \in \mathrm{Irr}_p(G)} \chi(1) \right)^{1/|\mathrm{Irr}_p(G)|}$$

so that $\mathrm{gm}_p(G)$ is the geometric mean degree of linear characters and irreducible characters of G with degree divisible by p.

We predict that

if G is a finite group such that $\mathrm{gm}_p(G) < \mathrm{gm}(H) = (\alpha(p)p)^{1/(\alpha(p)p+1)}$
then G has a normal Sylow p-subgroup.

This, if true, would improve the celebrated Itô-Michler theorem [8, 11] stating that if the degree of every irreducible character of a finite group G is coprime to p, then G has a normal Sylow p-subgroup.

Similarly, when p is an odd prime, the dihedral group D_{2p} is the smallest group without a normal p-complement, and when $p = 2$, the smallest group without a normal 2-complement is the alternating group A_4. Notice that $\mathrm{gm}(\mathsf{A}_4) = 3^{1/4}$ and $\mathrm{gm}(D_{2p}) = 2^{(p-1)/(p+3)}$. Let $\mathrm{gm}_{p'}(G)$ denote the geometric mean of character degrees corpime to p of a group G. That is,

$$\mathrm{gm}_{p'}(G) := \left(\prod_{\chi \in \mathrm{Irr}_{p'}(G)} \chi(1) \right)^{1/|\mathrm{Irr}_{p'}(G)|},$$

where

$$\mathrm{Irr}_{p'}(G) := \{\chi \in \mathrm{Irr}(G) \mid p \nmid \chi(1)\}.$$

Following the ideas in [4, 9], we predict that

> if G is a finite group and p a prime such that $\mathrm{gm}_{p'}(G) < 3^{1/4}$ when $p = 2$ and $\mathrm{gm}_{p'}(G) < 2^{(p-1)/(p+3)}$ when $p > 2$, then G has a normal p-complement.

This would improve Thompson's theorem [13] on character degrees stating that if the degree of every nonlinear irreducible character of a finite group G is divisible by p, then G has a normal p-complement.

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