ALBANIAN JOURNAL OF MATHEMATICS
Volume 14, Number 1, Pages 79–90
ISSN: 1930-1235; (**2020**)

ON THE HILBERT FUNCTION OF INTERSECTIONS OF A HYPERSURFACE WITH GENERAL REDUCIBLE CURVES

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Abstract. Let $W \subset \mathbb{P}^n$, $n \ge 3$, be a degree k hypersurface. Consider a " general " nodal union of d lines $L_1, \dots L_d$ with $L_i \cap L_j \ne \emptyset$ if and only if $|i-j| \le 1$ (here called a degree d bamboo). We study the Hilbert function of the set $Y \cap W$ with cardinality $k\deg(Y)$ and prove that it is the expected one (with a few classified exceptions (n,k,d)) when W is either a quadric hypersurface of rank at least 2 or $n = 3$ and W is an integral cubic surface.

Mathematics Subject Classes 2010: 14H50
Keywords: curves in projective spaces; lines; Hilbert function; union of lines

1. Introduction

For any positive integer d a degree d *abstract bamboo* T is a nodal projective curve T with d irreducible components, all of them smooth and of genus 0, such that there is an ordering $L_1, \dots, L_d$ of the irreducible components of T such that $L_i \cap L_j \ne \emptyset$ if and only if $|i-j| \le 1$ and $\#(L_i \cap L_{i+1}) = 1$ for $i = 1, \dots, d-1$. Any such ordering will be called a *good ordering*. Any abstract bamboo is connected and with arithmetic genus 0. Let $A(d)$ denote the set of all degree d abstract bamboos. Each $T \in A(d)$ is connected, nodal and $p_a(T) = 0$. For all integers $n \ge 2$, $d > 0$ let $A(n,d)$ denote the set of all pairs (T,f), where $T \in A(d)$ and $f : T \longrightarrow \mathbb{P}^n$ is a morphism such that $f(T)$ has at most nodes and it is the union of d distinct lines. The sets $A(d)$ and $A(n,d)$ are irreducible and non-empty. If $n \ge 3$ the map $f : T \longrightarrow \mathbb{P}^n$ is an embedding for a general $(T,f) \in A(n,d)$. For any $n \ge 3$ let $B(n,d)$ denote the set of all $f(T)$ with $(T,f) \in A(n,d)$ and $f : T \longrightarrow \mathbb{P}^n$ an embedding. We call *bamboos* or degree d bamboos of $\mathbb{P}^n$ the elements of $B(n,d)$.

Fix a reduced hypersurface $W \subset \mathbb{P}^n$. Let $A(n,d,W)$ denote the set of all $(T,f) \in A(n,d)$ such that the degree d nodal curve $f(T)$ is transversal to W, i.e. no irreducible component of $f(T)$ is contained in W and the set $f(T) \cap W$ is

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The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

formed by $d\deg(W)$ points, all of them smooth points of W. The set $A(n,d,W)$ is a nonempty open subset of $A(n,d)$. If $n\ge 3$ let $B(n,d,W)$ denote the set of all $T\in B(n,d)$ transversal to W.

Let $M\subseteq \mathbb{P}^n$ be an irreducible variety. Let $E\subset M$ be any closed subscheme. We will say that E has *maximal rank in* M if for each $t\in \mathbb{N}$ the restriction map $H^0(\mathcal{O}_M(t))\longrightarrow H^0(\mathcal{O}_E(t))$ has maximal rank, i.e. it is injective or surjective. If E is a finite set, E has maximal rank in M if and only if $h^0(M,\mathcal{I}_{E,M}(t)) = \max\{0,h^0(\mathcal{O}_M(t))-\#E\}$ for all t. Fix $\mathcal{L}\in \mathrm{Pic}(M)$. We will say that E has *maximal rank with respect to* $\mathcal{L}$ if the restriction map $H^0(M,\mathcal{L})\longrightarrow H^0(E,\mathcal{L}_{|E})$ has maximal rank.

Why reducible curve and in particular why reducible curves whose irreducible components are lines, e.g. bamboos ? They are useful even if one is only interested in the study of smooth curves in projective spaces ([1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 16, 17]). Why a few mathematicians care about the Hilbert function of the intersection of a curve $C\subset\mathbb{P}^n$ with a hypersurface of $\mathbb{P}^n$, often a quadric for $n=3$ and a hyperplane for $n>3$ (see the string of papers just quoted and [13, 14, 15, 18, 19]) ? Because it is often a key step to prove that C has the expected postulation. We also point out that proving that the general element of a tiny family of curves has nice intersection with a hypersurface may be very efficiently use for results on smooth curves ([2, 6]).

We prove the following result.

Theorem 1. *Let $W\subset\mathbb{P}^3$ be an integral cubic surface. Fix a positive integer d. For a general $X\in B(3,d,W)$ the set $X\cap W$ has maximal rank in W for the line bundle $\mathcal{O}_W(t)$ except in the following cases:*

$$(t,d)\in\{(1,1),(1,2),(2,2),(2,3),(2,4)\}.$$

Remark 7 explains the exceptional cases and computes their cohomology groups. These are exceptional cases for all degree d connected and reduced curves transversal to W. A key step of the proof of Theorem 1 (Lemma 9) works with minimal modifications if $W\subset\mathbb{P}^3$ is an integral surface with arbitrary degree. However, for each integer $\deg(W)\ge 4$ one should expect a few exceptional cases and we have no idea on them and how to do the initial cases to start the inductive proof.

We also prove the following result.

Theorem 2. *Fix integers $d\ge n\ge 2$. Let $W\subset\mathbb{P}^n$ be a quadric hypersurface of rank $\rho\ge 2$. There is $(T,f)\in A(n,d,W)$ such that the set $f(T)\cap W$ has maximal rank.*

Acknowledgments: We thank a referee for several key suggestions and important observations.

2. Preliminaries and the proof of Theorem 2

Fix a positive integer d. Let T be either an element of $A(n,d)$ or, if $n\ge 3$, an element of $B(n,d)$. By the definitions of $A(n,d)$ and $B(n,d)$ there is a good ordering $L_1,\dots,L_d$ of the irreducible components of T. If $d=1$ we say that T is a *final line* of T. If $d\ge 2$ we say that L_1 and L_d are the final lines of T. For $d\ge 2$ an irreducible component of T is a final line if and only if it meets only another irreducible components of T.

In this section we fix 2 different hyperplanes, U and V, of $\mathbb{P}^n$, $n \geq 2$, and set $W := U \cup V$.

Let $H \subset \mathbb{P}^n$ be a general hyperplane. Set $D := W \cap H$. For any integer $t \geq 0$ we have $h^0(\mathcal{O}_W(t)) = \binom{n+t}{n} - \binom{n+t-2}{n}$. Thus for all integers $t > 0$ we have $h^0(\mathcal{O}_W(t)) - h^0(\mathcal{O}_W(t-1)) = h^0(\mathcal{O}_D(t)) = \binom{n+t-1}{n-1} - \binom{n+t-3}{n-1}$.

Note that for any $(T, f) \in A(n, d, W)$ the integer $\deg(f(T) \cap W) = 2d$ is even. For all integers $n \geq 2$ and $t \geq 1$ we define the following Assertion $R(n, t)$:

Assertion $R(n,t)$: There is $(T, f) \in A(n, \lfloor h^0(\mathcal{O}_W(t))/2 \rfloor, W)$ such that $h^1(W, \mathcal{I}_{f(T)\cap W,W}(t)) = 0$.

Remark 1. By semicontinuity if $R(n, t)$ is true, then $h^1(W, \mathcal{I}_{h(Y)\cap W}(t)) = 0$ for a general $(Y, h) \in A(n, \lfloor h^0(\mathcal{O}_W(t))/2 \rfloor)$. If $n \geq 3$ it is sufficient to find $B \in B(n, \lfloor h^0(\mathcal{O}_W(t))/2 \rfloor)$ such that W contains no irreducible component of B and $h^1(W, \mathcal{I}_{W\cap B}(t)) = 0$.

Remark 2. $R(2, t)$ is true for all t, because $f(T)$ is a general union of d lines of $\mathbb{P}^2$ for a general $(T, f) \in A(2, d)$.

Remark 3. We claim that $R(n, 1)$ is true if and only if $n = 2$. $R(2, 1)$ is true (Remark 2). Assume $n \geq 3$ and take a general $(T, f) \in A(n, x)$, $x := \lfloor (n + 1)/2 \rfloor$. The curve $f(T)$ spans a linear space of dimension x. Hence $f(T) \cap W$ is a union of $2x$ points spanning a linear subspace of dimension $\leq x$. Since $2x \geq x + 2$, $R(n, 1)$ fails.

Remark 4. Fix integers $n \geq 3$, $t \geq 2$, and assume that $R(n, t)$ is true. Set $d := \lfloor h^0(\mathcal{O}_W(t))/2 \rfloor$ and take a general $B \in B(n, d)$. The curve B is transversal to W, $B \cap U$ is a general union of d points of U, $B \cap V$ is a general union of d points of V and $h^1(W, \mathcal{I}_{W\cap B,W}(t)) = 0$. If $h^0(\mathcal{O}_W(t))$ is even, then $h^0(\mathcal{I}_{W\cap B,W}(t)) = 0$. Now assume $h^0(\mathcal{O}_W(t))$ odd and hence $h^0(\mathcal{I}_{W\cap B,W}(t)) = 1$. Let E denote the zero-locus of any $f \in H^0(W, \mathcal{I}_{B\cap W,W}(t)) \setminus \{0\}$. Fix a good ordering $L_1, \dots, L_d$ of the irreducible components of B. Thus L_d a final line of B. Fix a general $o \in L_d$. Thus $o \notin L_d \setminus L_d \cap L_{d-1}$ and $o \notin W$. For a general $p \in U$ let D_p denote the line spanned by $\{o, p\}$. Set $B_p := B \cup L_d$. Note that $B_p \in B(n, d + 1)$ and that B_p is transversal to W. For a general p obviously $B_p \cap U$ is a general union of $d + 1$ points of U. Since $o \notin W$ and p is general in o it is easy to check that $D_p \cap V$ is a general point of V. Thus $B_p \cap V$ is a general union of $d + 1$ points of V. Hence $D_p \cap W \nsubseteq E$. Thus $h^0(W, \mathcal{I}_{B_p\cap W,W}(t)) = 0$.

Lemma 1. *Fix integers $n \geq 3$ and $d > 0$ and take a general $B \in B(n, d)$. Then either $h^1(\mathcal{I}_B(2)) = 0$ or $h^0(\mathcal{I}_B(2)) = 0$.*

Proof. First assume $d \leq n$. Since B is general, B spans a linear space M of dimension d. By induction on d we easily see that $h^1(M, \mathcal{I}_{B,M}(t)) = 0$ for all $t \in \mathbb{N}$. Hence $h^1(\mathcal{I}_B(t)) = 0$ for all t. Thus we may assume $d > n$. (a) Assume $n = 3$. First assume $d = 4$. Take a good ordering L_1, L_2, L_3, L_4 of the lines of B. Let H_1 be the plane spanned by $L_1 \cup L_2$ and H_2 the plane spanned by $L_3 \cup L_4$. To prove the lemma in this case it is sufficient to prove that $|\mathcal{I}_{B,\mathbb{P}^3}(2)| = \{H_1 \cup H_2\}$. Fix $Q \in |\mathcal{I}_{B,\mathbb{P}^3}(2)|$. $Q \cap H_2$ contains the conic $L_3 \cup L_4$ and the point $L_1 \cap H_2 \notin L_1 \cup L_2$. Thus $H_2 \subset Q$. In the same way we see that H_1 is an irreducible component of Q, concluding this case. Now assume $d = 5$. Take $L_1 \cup L_2 \cup L_3 \cup L_4$ as in the case $d = 4$ and call R a general line meeting L_4. Since $R \nsubseteq H_1 \cup H_2$, $h^0(\mathcal{I}_{L_1\cup L_2\cup L_3\cup L_4\cup R}(2)) =$

0, concluding the proof of this case. The case $d > 5$ follows from the case $d = 5$, because any degree $d > 5$ bamboo contains a degree 5 bamboo.

(b) Assume $n \geq 4$. Fix a hyperplane $H \subset \mathbb{P}^n$. Fix a general $A \in B(n, n)$, call L_n one of its final lines and set $\{o\} := L_n \cap H$. Let $T \subset H$ be a general bamboo of degree $d - n$ with the only restriction that $o \notin \mathrm{Sing}(T)$ and o is contained in a final line of T. Thus $A \cup T$ may be used to prove the lemma. Consider the residual exact sequence of H in $\mathbb{P}^n$:

$$0 \longrightarrow \mathcal{I}_A(1) \longrightarrow \mathcal{I}_{A\cup T}(2) \longrightarrow \mathcal{I}_{(A\cap H)\cup T,H}(2) \longrightarrow 0 \tag{1}$$

Since $h^i(\mathbb{P}^n, \mathcal{I}_A(1)) = 0$, $i = 0, 1$, and $(A \cup T) \cap H$ is a general union of T and $n - 1$ points of H, the lemma follows from induction on n.

□

Lemma 2. *$R(n, 2)$ is true for all $n \geq 3$.*

Proof. Fix a bamboo $T \subset \mathbb{P}^n$ transversal to W. Consider the exact sequence

$$0 \longrightarrow \mathcal{I}_{T,\mathbb{P}^n} \longrightarrow \mathcal{I}_{T,\mathbb{P}^n}(2) \longrightarrow \mathcal{I}_{T\cap W,W}(2) \longrightarrow 0 \tag{2}$$

Since $h^1(\mathcal{I}_{T,\mathbb{P}^n}) = 0$ and $h^2(\mathcal{I}_{T,\mathbb{P}^n}(2)) = h^1(\mathcal{O}_T) = 0$, the long cohomology exact sequence of (2) gives that $h^1(W, \mathcal{I}_{T\cap W,W}(2)) = 0$ if and only if $h^1(\mathbb{P}^n, \mathcal{I}_{T,\mathbb{P}^n}(2)) = 0$ and a similar statement holds for h^0. Thus to prove the lemma it is sufficient to use that either $h^1(\mathbb{P}^n, \mathcal{I}_{T,\mathbb{P}^n}(2)) = 0$ or $h^0(\mathbb{P}^n, \mathcal{I}_{T,\mathbb{P}^n}(2)) = 0$ (Lemma 1).

□

Lemma 3. *Fix integers $n \geq 2$ and $t \geq 3$. Assume $R(n, t-1)$. If $n \geq 3$ assume $R(n-1, t)$. Then $R(n, t)$ is true.*

Proof. Since the case $n = 2$ is true by Remark 2, we may assume $n \geq 3$ and use induction on n. Set $x := \lfloor h^0(\mathcal{O}_W(t))/2 \rfloor$ and $y := \lfloor h^0(\mathcal{O}_W(t-1))/2 \rfloor$.

(a) Assume that either $h^0(\mathcal{O}_W(t-1))$ is even or $h^0(\mathcal{O}_W(t))$ is odd. Note that we are looking for bamboos of degree x. Fix a general $Y \in B(n, y)$. Note that $2(x - y) \leq h^0(\mathcal{O}_{W\cap H}(t)) \leq 2(x - y) + 1$. Since H is general, H is transversal to Y, $Y \cap H \cap W = \emptyset$ and $H \cap W$ is a rank 2 quadric hypersurface of H. Fix $o \in H \cap Y$ with o in a final line of Y. Since the group of all $g \in \mathrm{Aut}(H)$ such that $g(H \cap W) = H \cap W$ acts transitively on $H \setminus H \cap W$, by the inductive assumption (case $n \geq 4$) or by Remark 2 (case $n = 3$) there is $(T, f) \in A(n-1, x-y, W \cap H)$ such that o is a smooth point of $f(T)$ belonging to a final line of $f(T)$, $f(T)$ is transversal to $W \cap H$ and $h^1(W \cap H, \mathcal{I}_{W\cap H\cap f(T)}(t)) = 0$. Set $B := Y \cup T$. If $n \geq 4$, f is an embedding and $Y \cup T \in B(n, x)$. If $n = 3$ there is $h : B \longrightarrow \mathbb{P}^3$ with $h_{|Y}$ the identity map and $h_{|T} = f$. Thus we may use B to test $R(n, t)$. Consider the residual exact sequence of $H \cap W$ in W:

$$0 \longrightarrow \mathcal{I}_{Y,W}(t-1) \longrightarrow \mathcal{I}_{B,W}(t) \longrightarrow \mathcal{I}_{f(T)\cap W\cap H,W\cap H}(t) \longrightarrow 0 \tag{3}$$

Since $h^1(W, \mathcal{I}_{Y,W}(t-1)) = 0$ and $h^1(H \cap W, \mathcal{I}_{f(T)\cap W\cap H,W\cap H}(t)) = 0$, (3) gives $h^1(W, \mathcal{I}_{B,W}(t)) = 0$.

(b) Assume $h^0(\mathcal{O}_W(t-1))$ odd and $h^0(\mathcal{O}_W(t))$ even. Take a solution Y of $R(n, t-1)$ with L_1 and L_y its final lines. Note that $y \geq 2$ and so $L_1 \neq L_y$. Fix a general line D intersecting L_y and set $\{u, v\} := D \cap W$ with $u \in U$ and $v \in V$. Note that $x - y \geq 2$. Let $H \subset \mathbb{P}^n$ be a general hyperplane containing v. Set $\{o\} := L_1 \cap H$. As in step (a) we see the existence of $(T, f) \in A(n-1, x-y-1)$ containing o and with $h^1(W \cap H, \mathcal{I}_{W\cap H\cap f(T), W\cap H}(t)) = 0$. We take $B := Y \cup D \cup f(T)$ as image of some $(B', h) \in A(n, x, W)$ satisfying $R(n, t)$.

□

Proof of Theorem 2: By semicontinuity it is sufficient to prove the theorem for a rank 2 quadric, say $W = U \cup V$ with U and V hyperplanes. Since the case $n = 2$ is true (Remark 2) we may assume $n \geq 3$. For all $t \in \mathbb{N}$ set $x_t := \lfloor h^0(\mathcal{O}_W(t)/2\rfloor$. Thus $x_1 = \lfloor (n+1)/2\rfloor$ and $x_{t-1} < x_t$ for all integers $t \geq 2$. Since $d \geq n$, there is a unique integer t such that $x_{t-1} < d \leq x_t$. Since $R(n, t)$ is true (Lemma 2 for $t = 2$ and Lemma 3 for $t \geq 3$) a general $X_t \in B(n, x_t)$ is transversal to W and $h^1(W, \mathcal{I}_{X_t\cap W, W}(t)) = 0$. Take any bamboo $B \subseteq X_t$ with $\deg(B) = d$. Since $B \cap W \subseteq X_t \cap W$, $h^1(W, \mathcal{I}_{B\cap W,W}(t)) = 0$. The Castelnuovo-Mumford's Lemma gives $h^1(W, \mathcal{I}_{B\cap W,W}(x)) = 0$ for all $x > t$. If $t = 2$ we get $h^0(W, \mathcal{I}_{B\cap W,W}(1)) = 0$ by the assumption $d \geq n$ which implies that B spans $\mathbb{P}^n$. Thus if $t = 2$ $B \cap W$ has maximal rank in W. Assume $t \geq 3$. Thus we may use $R(n, t-1)$. Since X_t is general, any bamboo $E \subset B$ is general in $B(n, \deg(E))$. Take a bamboo $X_{t-1} \subset B$ with $\deg(X_{t-1}) = x_{t-1}$. $R(n, t-1)$ gives $h^0(W, \mathcal{I}_{X_{t-1}\cap W,W}(t-1)) \leq 1$. B may be considered as a general bamboo obtained from X_{t-1} adding at least one line. Remark 4 gives $h^0(W, \mathcal{I}_{B\cap W,W}(t-1)) = 0$. Thus $B \cap W$ has maximal rank in W.

□

3. Proof of Theorem 1

In this section we work in $\mathbb{P}^3$ and prove Theorem 1. Let $W \subset \mathbb{P}^3$ be an integral cubic surface. Since $h^1(\mathcal{O}_{\mathbb{P}^3}(t-3)) = 0$ for all $t \in \mathbb{Z}$ and $h^2(\mathcal{O}_{\mathbb{P}^n}(t-3)) = 0$ for all $t \geq 0$, a standard exact sequence gives $h^0(\mathcal{O}_W(t)) = \binom{t+3}{3} - \binom{t}{3}$ and $h^1(\mathcal{O}_W(t-2)) = 0$ for all $t \geq 1$. Set $x_t := \lfloor h^0(\mathcal{O}_W(t))/3\rfloor$. Note that $h^0(\mathcal{O}_W(t)) \equiv 1 \pmod 3$ for all $t \geq 0$, i.e. $h^0(\mathcal{O}_W(t)) = 3x_t + 1$ for all $t \geq 0$.

Remark 5. For any $Y \in B(3, d, W)$ the scheme $Y \cap W$ is the union of $3d$ points. Thus $Y \cap W$ has maximal rank in W if and only if $h^0(W, \mathcal{I}_{W\cap Y,W}(t)) = 0$ for all t such that $x_t < d$ and $h^1(W, \mathcal{I}_{W\cap Y,W}(t)) = 0$ for all t such that $d \leq x_t$.

Consider the following assertion $H(t)$, $t \geq 3$:

Assertion $H(t)$, $t \geq 3$: $h^1(W, \mathcal{I}_{W\cap Y,W}(t)) = 0$ for a general $Y \in B(3, x_t, W)$.

Note that Assertion $H(t)$ is true if and only if $h^0(W, \mathcal{I}_{W\cap Y,W}(t)) = 1$ for a general $Y \in B(3, x_t)$. By the semicontinuity theorem for cohomology $H(t)$ is true if and only if $h^1(W, \mathcal{I}_{W\cap Y,W}(t)) = 0$ for at least one $B \in B(3, x_t, W)$.

The following example shows our main reason for using a quadric instead of a plane for an inductive proof of Theorem 1 and of $H(t)$ for $t \geq 5$.

Example 1. Let $M \subset \mathbb{P}^2$ be an integral plane cubic. For each $(T, f) \in A(2, d, M)$ the scheme $f(T)$ is a degree d plane curve. Thus the scheme $f(T) \cap M$ is the complete intersection of a plane cubic and a degree d curve. Hence it has not maximal rank, but it is very near to having it. Indeed, $h^0(\mathcal{O}_M(k)) = 3k$ for all $k > 0$. Note that $f(T) \cap M \in |\mathcal{O}_M(d)|$. The cohomology of line bundles of the integral curve genus 1 curve M gives that the restriction map $\rho_k : H^0(\mathcal{O}_M(k)) \longrightarrow H^0(\mathcal{O}_{M\cap f(T)}(k))$ is surjective if $k > d$, injective if $k < d$, while $\dim\ker(\rho_d) = \dim\mathrm{coker}(\rho_d) = 1$.

Remark 6. Fix $T \in B(3, 3)$. Since $\deg(T) = 3$ and $p_a(T) = 0$, T spans $\mathbb{P}^3$. Thus $h^1(\mathcal{I}_T(1)) = 0$. Since $h^2(\mathcal{I}_T) = h^1(\mathcal{O}_T) = 0$, the Castelnuovo-Mumford's Lemma gives $h^1(\mathcal{I}_T(t)) = 0$ for all $t \geq 1$.

Lemma 4. *Fix a general $T \in B(3,4)$. Then $h^1(\mathcal{I}_T(t)) = 0$ for all $t \geq 2$.*

Proof. By the Castelnuovo-Mumford's Lemma it is sufficient to prove the case $t = 2$.

Take two different planes H and M. Fix general lines L_1, L_2 of H and a general line L_4 of M. Let $L_3 \subset M$ a general line containing the point $L_2 \cap M$. Note that $T := L_1 \cup L_2 \cup L_3 \cup L_4$ is a bamboo contained in $H \cup M$. Take any $W \in |\mathcal{I}_T(2)|$. Since $W \cap M$ contains $L_3 \cup L_4 \cup \{L_1 \cap M\}$, M is an irreducible component of W. Similarly, $H \subset W$. Thus $|\mathcal{I}_T(2)| = \{H \cup M\}$.

□

Lemma 5. *Fix a general $Y \in B(3,5,W)$. We have $h^0(W, \mathcal{I}_{W\cap Y,W}(x)) = 0$ for all $x \leq 2$ and $h^1(W, \mathcal{I}_{W\cap Y,W}(t)) = 0$ for all $t \geq 3$.*

Proof. Since $W \cap Y$ spans $\mathbb{P}^3$, $h^0(W, \mathcal{I}_{W,\mathbb{P}^3}(1)) = 0$. Fix a general $E \in B(3,4)$. Lemma 4 gives $h^0(\mathbb{P}^3, \mathcal{I}_{E,\mathbb{P}^3}(2)) = 1$. Adding a line to E we get a degree 5 bamboo Y such that $h^0(\mathbb{P}^3, \mathcal{I}_{Y,\mathbb{P}^3}(2)) = 0$. Since $h^1(\mathbb{P}^3, \mathcal{I}_{Y,\mathbb{P}^3}) = 0$, the residual exact sequence of W gives $h^0(W, \mathcal{I}_{W\cap Y,W}(2)) = 0$.

Claim: $h^1(\mathbb{P}^3, \mathcal{I}_{Y,\mathbb{P}^3}(3)) = 0$.

Proof of the Claim: Let $Q \subset \mathbb{P}^3$ be a smooth quadric. Take 3 distinct elements L_1, L_3, L_5 of $|\mathcal{O}_Q(1,0)|$. Let L_2 be a general line of $\mathbb{P}^3$ intersecting both L_1 and L_3. Let L_4 be a general line of $\mathbb{P}^3$ intersecting both L_5 and L_3. Set $F := L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5$. Since $F \in B(3,5)$, by semicontinuity to prove the claim it is sufficient to prove that $h^1(\mathbb{P}^3, \mathcal{I}_{F,\mathbb{P}^3}(3)) = 0$. Note that $F \cap Q = L_1 \cup L_3 \cup L_5$ scheme-theoretically. Thus the residual exact sequence of Q gives the following exact sequence

$$0 \longrightarrow \mathcal{I}_{L_2\cup L_4,\mathbb{P}^3}(1) \longrightarrow \mathcal{I}_{F,\mathbb{P}^3}(3) \longrightarrow \mathcal{I}_{L_1\cup L_3\cup L_5,Q}(3) \longrightarrow 0 \tag{4}$$

Use $h^1(\mathbb{P}^3, \mathcal{I}_{L_2\cup L_4,\mathbb{P}^3}(1)) = 0$, $h^1(Q, \mathcal{I}_{L_1\cup L_3\cup L_5,Q}(3)) = h^1(Q, \mathcal{O}_Q(0,3)) = 0$ and the cohomology exact sequence of (4).

By the Castelnuovo-Mumford's Lemma to prove the h^1-vanishing it is sufficient to prove that $h^1(W, \mathcal{I}_{Y\cap W}(3)) = 0$, which is true by the residual exact sequence of W, because $h^1(\mathbb{P}^3, \mathcal{I}_{Y,\mathbb{P}^3}) = h^1(\mathbb{P}^3, \mathcal{I}_{Y,\mathbb{P}^3}(3)) = 0$.

□

Lemma 6. *There is a degree 6 bamboo $T \subset \mathbb{P}^3$ such that $h^1(\mathcal{I}_T(3)) = 0$.*

Proof. Fix a plane $H \subset \mathbb{P}^3$ and a general reducible conic $L_5 \cup L_6 \subset H$. Fix a general $o \in L_4$. Let $Y := L_1 \cup L_2 \cup L_3 \cup L_4$ be a general degree 4 bamboo containing o. Thus $T := Y \cup L_5 \cup L_6$ is a degree 6 bamboo. Consider the residual exact sequence of H:

$$0 \longrightarrow \mathcal{I}_Y(2) \longrightarrow \mathcal{I}_T(3) \longrightarrow \mathcal{I}_{T,H}(3) \longrightarrow 0 \tag{5}$$

Since $T \cap H$ is the union of $L_5 \cup L_6$ and 3 general points of H, $h^0(H, \mathcal{I}_{T\cap H,H}(3)) = 0$, i.e. $h^1(H, \mathcal{I}_{T\cap H,H}(3)) = 0$. Lemma 4 gives $h^1(\mathcal{I}_Y(2)) = 0$. Use the long cohomology exact sequence of (5).

□

Lemma 7. *Let $W \subset \mathbb{P}^3$ be any degree 3 surface (even reducible or with multiple components). Let $Y \subset \mathbb{P}^3$ be a general degree 6 bamboo and $X \subset \mathbb{P}^3$ be a general* *degree* 7 *bamboo. Then* $\dim Y \cap W = \dim X \cap W = 0$, $h^1(W, \mathcal{I}_{Y\cap W,W}(3)) = 0$ *and* $h^0(W, \mathcal{I}_{W\cap X,W}(3)) = 0$.

Proof. Since Y and X are general, none of their irreducible components is contained in W. Thus the residual exact sequence of W gives the exact sequence

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{I}_Y(3) \longrightarrow \mathcal{I}_{W\cap Y,W}(3) \longrightarrow 0 \tag{6}$$

Lemma 6 gives $h^1(\mathcal{I}_Y(3)) = 0$. Since $h^2(\mathcal{I}_Y) = h^1(\mathcal{O}_Y) = 0$, the long cohomology exact sequence of (6) gives $h^1(W, \mathcal{I}_{W\cap Y,W}(3)) = 0$, i.e. $h^0(W, \mathcal{I}_{W\cap Y,W}(3)) = 1$, say $\{D\} := |\mathcal{I}_{W\cap Y,W}(3)|$. Take as X the union of Y and a general line L intersecting a final line of T. Observe that $L \cap W \nsubseteq D$.
□

Remark 7. Fix a positive integer d and $Y \in B(3, d, W)$. Set $S := Y \cap W$. Thus $\#S = 3d$.

(a) Assume $d = 1$. S is formed by 3 collinear points. Thus $h^1(\mathbb{P}^3, \mathcal{I}_{S,\mathbb{P}^3}(1)) = 1$, $h^0(\mathbb{P}^3, \mathcal{I}_{S,\mathbb{P}^3}(1)) = 2$ and $h^1(\mathbb{P}^3, \mathcal{I}_{S,\mathbb{P}^3}(t)) = 0$ for all $t \ge 2$. Thus $h^1(W, \mathcal{I}_{S,W}(1)) = 1$, $h^0(W, \mathcal{I}_{S,W}(1)) = 2$ and $h^1(W, \mathcal{I}_{S,W}(t)) = 0$ for all $t \ge 2$.

(b) Assume $d = 2$. Thus S is the union of 6 coplanar points contained in a reducible plane conic and in no other conic. Thus $h^1(\mathbb{P}^3, \mathcal{I}_{S,\mathbb{P}^3}(1)) = 3$, $h^0(\mathbb{P}^3, \mathcal{I}_{S,\mathbb{P}^3}(1)) = 1$, $h^1(\mathbb{P}^3, \mathcal{I}_{S,\mathbb{P}^3}(2)) = 1$, $h^0(\mathbb{P}^3, \mathcal{I}_{S,\mathbb{P}^3}(2)) = 5$ and $h^1(\mathbb{P}^3, \mathcal{I}_{S,\mathbb{P}^3}(t)) = 0$ for all $t \ge 3$. Thus $h^1(W, \mathcal{I}_{S,W}(1)) = 3$, $h^0(W, \mathcal{I}_{S,W}(1)) = 1$, $h^1(W, \mathcal{I}_{S,W}(2)) = 1$, $h^0(W, \mathcal{I}_{S,W}(2)) = 5$ and $h^1(W, \mathcal{I}_{S,W}(t)) = 0$ for all $t \ge 3$.

(c) Assume $d = 3$. Obviously $h^0(\mathbb{P}^3, \mathcal{I}_{Y,\mathbb{P}^3}(1)) = 0$ and $h^1(\mathbb{P}^3, \mathcal{I}_{Y,\mathbb{P}^3}(t)) = 0$ for all $t > 0$ (Remark 6). Hence $h^0(\mathbb{P}^3, \mathcal{I}_{Y,\mathbb{P}^3}(2)) = 3$. Obviously $h^0(W, \mathcal{I}_{W\cap Y,W}(2)) \ge 3$. Since $h^1(\mathbb{P}^3, \mathcal{I}_{Y,\mathbb{P}^3}(-1)) = 0$, the residual exact sequence of W gives that $h^0(W, \mathcal{I}_{W\cap Y,W}(2)) = 3$. Thus $h^1(W, \mathcal{I}_{W\cap Y,W}(2)) = 2$. Since $h^1(\mathbb{P}^3, \mathcal{I}_{Y,\mathbb{P}^3}) = 0$, the residual exact sequence of W gives $h^0(W, \mathcal{I}_{Y\cap W,W}(3)) = h^0(\mathbb{P}^3, \mathcal{I}_{Y,\mathbb{P}^3}(3)) = 10$. Thus we have $h^1(W, \mathcal{I}_{Y\cap W,W}(3)) = 0$. Therefore the Castelnuovo-Mumford's Lemma gives that $h^1(W, \mathcal{I}_{Y\cap W,W}(t)) = 0$ for all $t \ge 3$.

(d) Assume $d = 4$. Lemma 4 gives $h^0(\mathbb{P}^3, \mathcal{I}_{Y,\mathbb{P}^3}(2)) = 1$. Since obviously $h^1(\mathbb{P}^3, \mathcal{I}_{Y,\mathbb{P}^3}(-1)) = 0$, the residual exact sequence of W gives $h^0(W, \mathcal{I}_{Y\cap W,W}(2)) = 1$. Thus $h^1(W, \mathcal{I}_{Y\cap W,W}(2)) = 3$.

Lemma 8. $H(3)$ *is true.*

Proof. Note that $x_3 = 6$. Apply Lemma 6. □

Lemma 9. *Fix an integer* $t \ge 5$ *and assume* $H(t-2)$. *Then* $H(t)$ *is true.*

Proof. For each $o \in \mathbb{P}^3$ let $\chi(o)$ denote the closed subscheme of $\mathbb{P}^3$ with $(\mathcal{I}_o)^2$ as its ideal sheaf. For any $S \subset \mathbb{P}^3$ such that $\#S = 2$ let $\langle S\rangle \subset \mathbb{P}^3$ denote the unique line containing S.

Since $3(t-1) \le \binom{t+1}{3} - \binom{t-2}{3}$ for all $t \ge 5$, $x_{t-2} \ge t-2$ for all $t \ge 5$. Since $H(t-2)$ is true, $h^1(W, \mathcal{I}_{B\cap W}(t-2)) = 0$ for a general $B \in B(3, x_{t-2}, W)$. Fix any such B. Since $\#\mathrm{Sing}(B) = x_{t-2} - 1$, $\#\mathrm{Sing}(B) \ge t-4$. Take any $S \subset \mathrm{Sing}(B)$ such that $\#S = t-4$ and set $Z := \cup_{o\in S}\chi(o)$ and $B' := B \cup Z$. Note that $B \setminus S$ has $t-3$ connected components and that the closure of each connected component of $B \setminus S$ is a bamboo. Call $U_1, \dots, U_{t-3}$ the closure of the connected components of $Y \setminus S$ and set $y_i := \deg(U_i)$. All positive integers $y_1, \dots, y_{t-3}$ such that $y_1 + \cdots + y_{t-3} = x_{t-2}$ appear for a certain S. By [10, Ex. 2.1.1] B' is a flat limit of a family of $t-3$ disjoint curves, each of them a bamboo, and with degrees $y_1, \dots, y_{t-3}$ and whose limits are bamboos $U_1, \dots, U_{t-3}$. Thus by semicontinuity for a general union $Y \subset \mathbb{P}^3$ of $t-3$ bamboos of degrees $y_1, \dots, y_{t-3}$ Y is transversal to W and $h^1(W, \mathcal{I}_{W \cap Y, W}(t)) = 0$. Call $Y_1, \dots Y_{t-3}$ the connected components of Y with $\deg(Y_i) = y_i$. In the following we only take $y_i = 1$ for $i = 1, \dots, t-4$, and $y_{t-2} = x_{t-2} - t + 4$ (this can be achieved for a suitable S). Thus Y_i is a line for $i \neq t-4$ and $Y_{t-3} \in B(3, x_{t-2} - t + 4)$. We call R_1 and R_2 the final lines of Y_{t-3} (we have $R_1 \neq R_2$, because $x_{t-2} > t-3$).

Let $Q \subset \mathbb{P}^3$ be a general smooth quadric. Set $D := Q \cap W$. Bertini's theorem gives that D is an integral element of $|\mathcal{O}_Q(3,3)|$. The adjunction formula gives $\omega_D \cong \mathcal{O}_D(1)$. D has arithmetic genus 4 and $h^0(\mathcal{O}_D(t)) = 6t-3$ for all $t \geq 2$. Since $h^1(\mathcal{O}_W(t-2)) = 0$ for all $t \geq 1$, $h^0(\mathcal{O}_W(t)) - h^0(\mathcal{O}_W(t-2)) = h^0(\mathcal{O}_D(t))$ for all $t \geq 2$. Note that $x_t = x_{t-2} + 2t - 1$. Fix $t+1$ general elements $A_1, \dots, A_{t+1}$ of $|\mathcal{O}_Q(1,0)|$ and $t-2$ general elements $B_1, \cdots B_{t-2}$ of $|\mathcal{O}_Q(0,1)|$.

Let $E \in |\mathcal{O}_Q(t+1, t-2)|$ be the union of all lines A_i and all lines B_j. Since $t \geq 5$, $t-2 \geq 3$. Set $E_1 := A_1 \cup B_1 \cup A_2$, $E_2 := A_3 \cup B_2 \cup A_4$ and $E_3 := A_5 \cup B_3 \cup A_6$. For $i = 4, \dots, t-2$ set $E_i := A_{i+3} \cup B_i$. Assume for the moment $t \geq 6$. Fix a general $o_1 \in A_2$, a general $o_2 \in A_3$, a general $o_3 \in A_4$, a general $o_4 \in A_5$ and a general $o_5 \in A_6$. For $i = 7, \dots, t+1$ take a general $p_i \in A_i$. For $i = 4, \dots, t-3$ let q_i be a general element of B_i. Note that we did not chose any $q_{t-2} \in B_{t-2}$. Write $S' := \{o_1, o_2, o_3, o_4, o_5, p_7, \dots, p_{t+1}, q_4, \dots, q_{t-3}\}$. Since the irreducible components of E are general and each of them contains at most one element of S', S' may be seen as a general subset of Q with cardinality $2t-6$. Note that $\#S'$ is twice the number of connected components of Y. Since a general line intersects transversally Q, for a general $S_1 \subset Q$ with $\#S_1 = 2$ the line $\langle S_1 \rangle$ is a general line of $\mathbb{P}^3$. Thus by semicontinuity we may assume the lines $\langle \{o_1, o_2\} \rangle$, $\langle \{o_3, o_4\} \rangle$, $\langle \{o_5, p_7\} \rangle$ and $\langle \{q_i, p_{i+4}\} \rangle$, $i = 4, \dots, t-4$, are $t-4$ general lines of $\mathbb{P}^3$. Thus by semicontinuity we may assume that these lines are $Y_1, \dots, Y_{t-4}$ (in this order). For any line $R \nsubseteq Q$ and any $o \in \mathbb{P}^3 \setminus R$ there is a line containing o and intersecting R at a general point. Thus we may assume that $p_{t+1} \in R_1$, that $q_{t-3} \in R_2$ and that neither p_{t+1} nor q_{t-3} is a singular point of Y_{t-3}.

For $t = 5$ Y has only 2 connected components, $E = E_1 \cup E_2 \cup E_3$ and we only take o_1, o_2, o_3, o_4. As in the case $t \geq 6$ we may assume that $Y_1 = \langle \{o_1, o_2\} \rangle$, that $o_3 \in R_1$ and that $o_4 \in R_2$. Set $J := Y \cup E$. Since the lines A_i and B_j are general, J is transversal to W.

Claim 1: $h^1(W, \mathcal{I}_{J \cap W, W}(t)) = 0$.

Proof of Claim 1: Note that $Y \cap Q \cap W = \emptyset$. The residual exact sequence of D in W shows that to prove Claim 1 it is sufficient to prove $h^0(D, \mathcal{I}_{E \cap D, D}(t,t)) = 0$. Since $E \in |\mathcal{O}_Q(t+1, t-2)|$, $D \cap E \in |\mathcal{O}_D(t+1, t-2)|$. Thus it is sufficient to prove that $h^0(D, \mathcal{O}_D(-1, 2)) = 0$. Use the exact sequence

$$0 \longrightarrow \mathcal{O}_Q(-4,-1) \longrightarrow \mathcal{O}_Q(-1,2) \longrightarrow \mathcal{O}_D(-1,2) \longrightarrow 0$$

and that $h^0(\mathcal{O}_Q(-1,2)) = h^1(\mathcal{O}_Q(-4,-1)) = 0$ by the Künneth formula.

Set $\Sigma := \mathrm{Sing}(E) \setminus (\mathrm{Sing}(E_1) \cup \cdots \cup \mathrm{Sing}(E_{t-2}))$. Note that Σ is the set of all $A_i \cap B_j$ for some i, j such that $A_i \cup B_j$ is not contained in the same curve E_h. Set $\chi := \cup_{o \in \Sigma} \chi(o)$. Since $\chi \cap W = \emptyset$, J is transversal to W and transversality is an open condition, Claim 1 shows that to conclude the proof of the lemma it is sufficient to prove the following Claim 2.

Claim 2: The scheme $J \cup \chi$ is in the closure of $B(3, x_t)$ in the Hilbert scheme of $\mathbb{P}^3$.

Proof of Claim 2: We use the notation of the case $t \geq 6$, the case $t = 5$ only requiring minimal modifications (see Proof of Claim 2 of Lemma 11 for the modifications needed for the case $t = 4$). We deform E and J in the following way. Each B_i remain fixed in the deformation and we keep fixed the points $q_1, \dots, q_{t-4}$ as linking points. Each line A_i, $1 \leq i \leq t+2$, is a flat limit of lines intersecting the only curve B_j such that $A_i \cup B_j$ is in the the same E_h, i.e. B_1 for A_1 and A_2, B_2 for A_3 and A_4, B_3 for A_5 and A_6 and B_i for A_{i+3} if $4 \leq i \leq t-2$. We call Δ a smooth and connected one-dimensional parameter space for these deformations, say $\pi : \mathcal{F} \longrightarrow \Delta$, with $0 \in \Delta$ such that $\pi^{-1}(0) = E$. For each $x \in \Delta$ call $A_i(x)$ the irreducible component $E(x) := \pi^{-1}(x)$ with A_i as its limit. Restricting if necessary Δ we may assume that for all $x \in \Delta \setminus \{0\}$ the lines $A_i(x)$ are pairwise disjoint and each of them intersects a unique B_h. Since each o_i and each p_i is a smooth point of E, up to an étale covering of Δ, we may assume that π has 5 sections O_i, $1 \leq i \leq 5$, and $t-5$ sections P_i, $7 \leq i \leq t+1$, such that $O_i(0) = o_i$ and $P_i(0) = p_i$ for all i. Restricting if necessary Δ we may assume that for each $x \in \Delta$ the values at x of these sections are different. For all $x \in \Delta$ we define the lines $Y_i(x)$, $1 \leq i \leq t-4$, in the following way. For each $x \in \Delta$ set $Y_1(x) := \langle \{o_1(x), o_2(x)\} \rangle$, $Y_2(x) := \langle \{o_3(x), o_4(x) \rangle$, $Y_3(x) := \langle \{o_5(x), p_7(x) \rangle$ and $Y_i := \langle \{q_i, p_{i+4}(x)\} \rangle$ for $4 \leq i \leq t-3$. We may deform Y_{t-3} to a bamboo $Y_{t-3}(x)$, $x \neq 0$, with final lines $R_1(x)$ and $R_2(x)$, with $R_1(x)$ containing q_{t-2} for all x and with $R_2(x)$ containing $p_{t+1}(x)$ for all x. Let $J(x)$ denote the union of $E(x)$ and these bamboos $Y_i(x)$, $1 \leq i \leq t-3$. Restricting if necessary Δ we may assume that these bamboos are pairwise disjoint and that they meet $E(x)$ only at the prescribed linking points. By [10, Ex. 2.1.1] $J \cup \chi$ is a flat limit of the family $J(x)$, $x \in \Delta \setminus \{0\}$. For $x \neq 0$, $J(x)$ is a connected and nodal curve with arithmetic genus 0. If we take as a good ordering of $Y_{t-3}(x)$ a good ordering with $R_1(x)$ as the first line and $R_2(x)$ as a last line the string of lines appearing in $E_1(x) \cup Y_1(x) \cup E_2(x) \cup Y_2(x) \cup \cdots \cup E_{t-3}(x) \cup Y_{t-3}(x) \cup E_{t-2}(x)$ is a good ordering of $J(x)$.

□

Lemma 10. *Let $Y \subset \mathbb{P}^3$ be a union of 3 disjoint lines, none of them contained in W. Then $h^0(W, \mathcal{I}_{Y \cap W, W}(2)) = 1$ and $h^1(W, \mathcal{I}_{Y \cap W, W}(2)) = 0$.*

Proof. Obviously Y is contained in a quadric, Q, which cannot be reducible or a cone, because Y has 3 connected components. Since Y is the union of 3 elements of a ruling of Q, $h^0(Q, \mathcal{I}_{Y,Q}(2)) = 0$ and hence Q is the unique quadric containing Y. Since no irreducible component of Y is contained in W and Y is a reduced curve, the residual exact sequence of W in $\mathbb{P}^3$ is the following exact sequence:

$$0 \longrightarrow \mathcal{I}_Y(-1) \longrightarrow \mathcal{I}_Y(2) \longrightarrow \mathcal{I}_{Y \cap W, W}(2) \longrightarrow 0 \tag{7}$$

Since Y is a reduced curve, $h^0(\mathcal{O}_Y(-1)) = 0$. Thus $h^1(\mathcal{I}_Y(-1)) = 0$. Since $h^0(\mathcal{I}_Y(2)) = 1$, (7) gives $h^0(W, \mathcal{I}_{Y \cap W, W}(2)) = 1$.

□

Lemma 11. *$H(4)$ is true.*

Proof. Note that $h^0(\mathcal{O}_W(4)) = 31$ and hence $x_4 = 10$. Thus we need to prove that $h^1(W, \mathcal{I}_{B \cap W, W}(4)) = 0$ for a general $B \in B(3, 10)$. We adapt the proof of Lemma 9 using the curve in Lemma 10 instead of a degree 3 bamboo.

Fix a general $(Y, Q) \in B(3,1)^3 \times |\mathcal{O}_{\mathbb{P}^3}(2)|$. Since Q is general, it is smooth. Set $D := Q \cap W$. Bertini's theorem gives that D is an integral element of $|\mathcal{O}_Q(3,3)|$. Call Y_1, Y_2, Y_3 the connected components of Y. Since (Y, Q) is general, Y is transversal to Q. Since 2 general points of Q spans a general line of $\mathbb{P}^3$, $Y \cap Q$ is a general subset of Q with cardinality 6. Fix $o_1 \in Y_1 \cap Q$ and set $\{o_2, o_3\} := Y_2 \cap Q$ and $\{o_4, o_5\} := Y_3 \cap Q$. Let A_i, $1 \le i \le 5$, the element of $|\mathcal{O}_Q(1,0)|$ containing o_i. Let B_1, B_2 be 2 general elements of $|\mathcal{O}_Q(0,1)|$. Set $E_1 := A_1 \cup B_1 \cup A_2$, $E_2 := A_3 \cup B_2 \cup A_4$, $E := E_1 \cup E_2 \cup A_3$ and $J := E \cup Y$. E_1 and E_2 are degree 3 bamboos and we take $A_1 \cup B_1 \cup A_2$ and $A_3 \cup B_2 \cup A_4$ as a good ordering of them.

Claim 1: $h^1(W, \mathcal{I}_{J \cap W}(4)) = 0$.

Proof of Claim 1: Lemma 10 gives $h^1(W, \mathcal{I}_{Y \cap W, W}(2)) = 0$. Note that $E \cap W = E \cap D \in |\mathcal{O}_D(5,2)|$. In the Proof of Claim 1 of Lemma 9 we proved that $h^1(D, \mathcal{I}_{E \cap W, D}(4,4)) = 0$. Hence $h^1(W, \mathcal{I}_{W \cap E, W}(4,4)) = 0$. Lemma 10 gives $h^1(W, \mathcal{I}_{Y \cap W, W}(2)) = 0$. Thus the residual exact sequence of the Cartier divisor D of W gives $h^1(W, \mathcal{I}_{J \cap W}(4)) = 0$.

The curve J is a connected nodal curve with degree 10 whose connected components are lines. Note that χ has 6 connected components, each of them with as its reduction a singular point of the nodal curve J. Since the nodal curve J is connected, with 10 smooth connected components and with 15 singular points, J has arithmetic genus 6. Thus $\chi(\mathcal{O}_{J \cup \chi}) = 0$ and $J \cup \chi$ and any $B \in B(3,10)$ have the same Hilbert polynomial. To prove $H(4)$ using Claim 1 and the semicontinuity theorem for cohomology it is sufficient to prove the following Claim 2.

Claim 2: The scheme $J \cup \chi$ is in the closure of $B(3,10)$ in the Hilbert scheme of $\mathbb{P}^3$.

Proof of Claim 2: A_1 and A_2 are flat limits of the family of all lines of $\mathbb{P}^3$ meeting B_1 and whose general member, say $\tilde{A}_1$ and $\tilde{A}_2$, is a general line of $\mathbb{P}^3$ meeting B_1. A_3 and A_4 are flat limits of the family of all lines of $\mathbb{P}^3$ meeting B_3 and whose general member, say $\tilde{A}_3$ and $\tilde{A}_4$, is a general line of $\mathbb{P}^3$ meeting B_2. A_5 is a line of $\mathbb{P}^3$ and hence it is a flat limit of the family of all lines of $\mathbb{P}^3$. Call $\tilde{A}_3$ a general line of $\mathbb{P}^3$. By [10, Ex. 2.1.1] $E \cup \chi$ is a flat limit of a flat family $\mathcal{F}$ whose general member is $\widetilde{E_1} \cup \widetilde{E_2} \cup \widetilde{A_1}$ with $\widetilde{E_1} := \widetilde{A_1} \cup B_2 \cup \widetilde{A_2}$ and $\widetilde{E_2} := \widetilde{A_3} \cup B_2 \cup \widetilde{A_4}$. Note that $\widetilde{E_1}$, $\widetilde{E_2}$ and $\widetilde{A_5}$ are pairwise bamboos of degree 3, 3 and 1 respectively. As expected by the good ordering of E_1 and E_2 we order the lines of $\widetilde{E_1}$ (resp. $\widetilde{E_2}$) so that $\widetilde{A_2}$ follows B_1 which follows $\widetilde{A_1}$ (resp. $\widetilde{A_2}$ follows B_1 which follows $\widetilde{A_1}$). We call $\pi : \mathcal{F} \longrightarrow \Delta$ the flat family and $0 \in \Delta$ the point such that $\pi^{-1}(0) = E \cup \chi$ and call z the general point of Δ such that $\pi^{-1}(z) = \widetilde{E_1} \cup \widetilde{E_2} \cup \widetilde{A_5}$. Since $\{o_1, o_2, o_3, o_4, o_5\}$ of in the smooth locus of $E \cup \chi$, up to an étale covering of Δ, we may assume the existence of 5 sections of π, say $s_i : \Delta \longrightarrow \mathcal{F}$, $1 \le i \le 5$, with $s_i(0) = o_i$ for all i. There is an algebraic way to chose for each $x \in \Delta$ a line $A_1(x)$ with $A_1(x) = s_1(x)$ for all $x \in \Delta$ and $A_1(z)$ the general line of $\mathbb{P}^3$ containing $s_1(z)$. Restricting if necessary Δ we may assume $\#\{s_1(x), s_2(x), s_3(x), s_4(x), s_5(x)\} = 5$ for all $x \in \Delta$. For each $x \in \Delta$ set $Y_2(x) := \langle \{s_2(x), s_3(x)\} \rangle$ and $Y_3(x) := \langle \{s_4(x), s_5(x)\} \rangle$. Thus $s_1(0) = Y_1$, $s_2(0) = Y_2$ and $s_3(0) = Y_3$. Since $Y_i \cap Y_j = \emptyset$ for all $i \ne j$, restricting if necessary Δ we may assume that $Y_i(x) \cap Y_i(x) = \emptyset$ for all $i \ne j$. We may also assume that for the general $x \in \Delta$ (and in particular for $x = z$) $Y_1(x)$ meets $E_1(x) \cup E_2(x) \cup A_5(x)$ only at $s_1(x)$, $Y_2(x)$ meets $E_1(x) \cup E_2(x) \cup A_5(x)$ only at

$\{s_2(x), s_3(x)\}$ and $Y_3(x)$ meets $E_1(x) \cup E_2(x) \cup A_5(x)$ only at $\{s_4(x), s_5(x)\}$. Thus $\pi^{-1}(z) := Y_1(z) \cup E_1(z) \cup Y_2(z) \cup E_2(z) \cup Y_3(z) \cup A_3(z)$ is a connected nodal curve with 10 connected components and 9 singular points. Thus $p_a(\pi^{-1}(z)) = 0$. To conclude the proof of Claim 2 and hence to conclude the proof of the lemma it is sufficient to fing a good ordering of the lines of $\pi^{-1}(z)$. The good ordering is the one coming from the list $Y_1(z) \cup E_1(z) \cup Y_2(z) \cup E_2(z) \cup Y_3(z) \cup A_3(z)$ with the good ordering of $E_1(z)$ and $E_2(z)$ obtained from the one of E_1 and E_2 adding the symbol ˜ to the lines A_i.

□

Proof of Theorem 1: We discussed in Remark 7 why the listed cases are exceptional and the amount of maximal rank failure for each of these cases. Thus to prove the theorem we may assume $d \geq 5$. For $d = 5$ use Lemma 5. For $d = 6$ use Lemma 7. From now on we assume $d \geq 7$. Let t be the minimal integer such that $x_t \geq d$. Thus $x_{t-1} < d \leq x_t$. Since $H(t)$ is true, there is $T \in B(3, x_t, W)$ such that $h^0(W, \mathcal{I}_{T\cap W}(t)) = 1$ and $h^1(W, \mathcal{I}_{T\cap W}(t)) = 0$. Thus $h^1(W, \mathcal{I}_{T\cap W}(x)) = 0$ for all $x \geq t$ by the Castelnuovo-Mumford's Lemma. Every connected subcurve of a bamboo is a bamboo. Let $E \subseteq T$ a connected degree d curve. Thus $h^1(W, \mathcal{I}_{W\cap E,W}(x)) = 0$ for all $x \geq t$. E is a bamboo. Since $B(3, d, W)$ is irreducible, it is sufficient to find $B \in B(3, d, W)$ such that $h^0(W, \mathcal{I}_{W\cap B,W}(t-1)) = 0$. By $H(t-1)$ there is $F \in B(3, x_{t-1}, W)$ such that $h^0(W, \mathcal{I}_{W\cap F,W}(t-1)) = 0$. Fix a final line L_1 of F and take $B := F \cup L$, where L is a general line intersecting L_1.

□

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