

THE HEAT KERNEL OF A WEIGHTED MAASS-LAPLACIAN
WITH REAL WEIGHTS

ZENAN ŠABANAC

*Department of Mathematics,
University of Sarajevo, Zmaja od Bosne 35,
71 000 Sarajevo*

LAMIJA ŠĆETA

*School of Economics and Business,
University of Sarajevo, Alije Izetbegovića 1,
71 000 Sarajevo*

ABSTRACT. We derive closed formula for the heat kernel $K_{\mathbb{H},k}$ associated to the Maass-Laplacian operator D_k for any real weight k and prove that the heat kernel $K_{\mathbb{H},k}$ is strictly monotone decreasing function of the hyperbolic distance. We derive small time and large time asymptotic formulae for the heat kernel $K_{\mathbb{H},k}$ and describe its behavior as a function of the real weight k .

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1. INTRODUCTION

Let $\Gamma \subset PSL(2, \mathbb{R})$ be a discontinuous subgroup of the group of all real Möbius transformations acting on the upper half-plane \mathbb{H} , and by $\hat{\Gamma}$ denote the group covering Γ under the projection $SL(2, \mathbb{R}) \rightarrow PSL(2, \mathbb{R})$. We assume that $-I \in \hat{\Gamma}$, where I denotes the identity element.

We assume that the fundamental domain of Γ has finite hyperbolic area, in which case the Riemann surface X_Γ identified with \mathbb{H}/Γ has a finite volume.

E-mail addresses: zsabanac@pmf.unsa.ba, lamija.sceta@efsa.unsa.ba.

The spectral theory of the Laplace operator and Laplace-like operators (Maass Laplacians) on Riemann surfaces is an important object of study with vast applications in theory of automorphic forms, string theory, scattering theory and differential geometry, to name a few.

The study of the determinant of the Laplacian acting on the space of twice continuously differentiable complex functions on a Riemann surface (actually, the study of its self-adjoint extension) is crucial in the Polyakov's string theory [19]. Spectral theory of the hyperbolic Laplacian

$$D_0 := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

and the related Selberg zeta function is a crucial ingredient in the construction of the determinant of the Lax-Phillips scattering operator, introduced in [17] (see [9] and the references therein).

The analytic and Reidemeister torsion on Riemannian manifolds is related to the spectral theory of the associated Laplacian, see [20], so the spectral theory of the Laplacian can be viewed as a tool to deduce geometric invariants.

When the manifold is compact, the study of the determinant of the Laplacian and the study of the analytic torsion and related invariants from the abovementioned references rely on the study of the spectral zeta function, defined as the Mellin transform of the corresponding heat kernel.

More recently, the fact that for any integral or half-integral k , the kernel of the operator $D_k - k(1 - k)$ is isomorphic to the space of weight $2k$ cusp forms for Γ (when both spaces are viewed as \mathbb{C} -vector spaces) was used in [7] and [8] to deduce effective sup-norm bounds on average for such cusp forms. Here D_k denotes the weighted Maass-Laplacian, defined by

$$(1.1) \quad D_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + 2iky \frac{\partial}{\partial x}.$$

The asymptotic behavior of the heat kernel of the Maass-Laplacian D_k on the upper half-plane, evaluated for $k \in \frac{1}{2}\mathbb{Z}$ in a closed form by [18] played a crucial role in the results of [7] and [8].

The operator D_k acts on twice continuously differentiable functions $f : \mathbb{H} \rightarrow \mathbb{C}$ satisfying transformation behavior (2.1) with respect to a certain multiplier system of weight k . Multiplier systems and weighted Maass-Laplacians can be defined for arbitrary real weights, not just for integral and half-integral weights, see the construction on pp. 335–337 of [12]. Moreover, there are examples of important multiplier systems with weight $k \notin \frac{1}{2}\mathbb{Z}$, constructed from generalized Dedekind sums associated to Fuchsian groups (see e.g. [2, 14, 3]). Therefore, it is of interest to deduce a closed formula for the heat kernel associated to the weighted Maass-Laplacian D_k on \mathbb{H} for any real k and establish its asymptotic behavior.

In this paper we derive a closed formula for the heat kernel associated to D_k for an arbitrary weight $k \neq 0$ (the case when $k = 0$ is well-known) and prove that it is strictly monotone decreasing when viewed as a function of the hyperbolic distance between the points in \mathbb{H} . We further investigate asymptotic behavior of the heat kernel as $t \downarrow 0$ and $t \rightarrow \infty$. We prove that the heat kernel decays exponentially as $t \downarrow 0$ and that the rescaled kernel is integrable in t on $[1, \infty)$. This result is important in various applications, e.g. one may use results of [13] and obtain a closed formula for a branched meromorphic continuation of the Poisson kernel

associated to D_k . Also, one may follow the pattern described in [16] to deduce meromorphic continuation of the spectral zeta function. We also investigate the behavior of the heat kernel as a function of a real weight k and prove that the heat kernel viewed as a function of k is strictly monotone decreasing when $k < 0$ and strictly monotone increasing when $k > 0$.

Since the heat kernel is a fundamental object in various mathematical disciplines, we are certain that a closed formula derived in this paper for all real weights k , together with description of its behavior as a function of the hyperbolic distance, as a function of time t , and as a function of the real weight k will have many applications.

2. PRELIMINARIES

Let $\mathbb{H} = \{z \mid z = x + iy, y > 0\}$ be the upper half-plane with Poincare metric $dz = \frac{|dz|}{y}$ and areal measure $d\mu(z) = \frac{dx dy}{y^2}$ expressed in the rectangular coordinates. By $r(z, w)$ we denote the hyperbolic distance between points $z, w \in \mathbb{H}$.

For $U = \begin{pmatrix} a_U & b_U \\ c_U & d_U \end{pmatrix} \in SL(2, \mathbb{R})$ and $z \in \mathbb{H}$ let $j(U, z) := c_U z + d_U$. Then, for any two matrices $U, T \in SL(2, \mathbb{R})$, there exist a unique number $w(U, T) \in \{-1, 0, 1\}$, the *phase factor* such that

$$2\pi i w(U, T) = -\log j(UT, z) + \log j(U, Tz) + \log j(T, z),$$

where logarithmic function on the right hand side is the principal branch normalized so that the argument takes values in $(-\pi, \pi]$, see e.g. [14, Section 2.2].

A (scalar-valued) *multiplier system* ν_k of a real weight k on $\hat{\Gamma}$ is a function $\nu_k : \hat{\Gamma} \rightarrow S^1$, where S^1 is a unit circle in \mathbb{C} with the properties:

- i) $\nu_k(-I) = \exp(-2\pi i k)$,
- ii) $\nu_k(UT) = \sigma_{2k}(U, T)\nu_k(U)\nu_k(T)$, for $U, T \in \hat{\Gamma}$,

where $\sigma_{2k}(U, T) := \exp(4\pi i k w(U, T))$ is the *factor system* of weight $2k$, see [6, Definition 1.3.1].

We denote by \mathfrak{F}_k the space of all functions $f : \mathbb{H} \rightarrow \mathbb{C}$, which transform as

$$(2.1) \quad f(Uz) = \nu_k(U) \left(\frac{cz + d}{c\bar{z} + d} \right)^k f(z),$$

for all $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \hat{\Gamma}$, where ν_k is the multiplier system of weight k on $\hat{\Gamma}$.

We denote by \mathfrak{H}_k the Hilbert space of all functions $f \in \mathfrak{F}_k$ that are square-integrable on the fundamental domain of Γ , with respect to the hyperbolic measure $d\mu(z)$.

The weighted Maass-Laplacian D_k , defined by (1.1) preserves the transformation behavior (2.1) for all twice continuously differentiable functions from \mathfrak{F}_k and acts on a dense subspace of \mathfrak{H}_k . It is proved in [21] that D_k extends to a self-adjoint operator on \mathfrak{H}_k .

Let us note here that weighted Maass-Laplacian D_k is a specialization of the operator

$$\Delta_{\alpha, \beta} = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + (\alpha - \beta)iy \frac{\partial}{\partial x} - (\alpha + \beta)y \frac{\partial}{\partial y}$$

at $\alpha + \beta = 0$, first studied by Maass in [15].

The heat kernel of the Maass-Laplacian D_k on the upper half-plane, denoted by $K_{\mathbb{H},k}(t; z, w)$, where $t > 0$ is real and $z, w \in \mathbb{H}$ is a unique fundamental solution to the differential operator $D_k + \partial_t$, satisfying a Dirac condition as $t \downarrow 0$:

$$f(z) = \lim_{t \downarrow 0} \int_{\mathbb{H},k} K_{\mathbb{H},k}(t; z, w) d\mu(w),$$

for any bounded and continuous function f on \mathbb{H} , where the convergence is uniform on compact subsets of \mathbb{H} (see [13]).

3. CLOSED FORMULA FOR THE HEAT KERNEL ASSOCIATED TO MAASS-LAPLACIAN OPERATOR D_k

In his seminal paper [5], Fay initiated the study of the heat kernel of D_k on \mathbb{H} , however, D’Hocker and Phong [4] established that this formula is not completely correct. In case when $k \in \frac{1}{2}\mathbb{Z}$, the correct expression for the heat kernel $K_{\mathbb{H},k}(t; z, w)$ of the Maass-Laplacian D_k on the upper half-plane was deduced by Oshima in [18].

It is well known fact that the heat kernel is a radial function, i.e. it depends only upon the hyperbolic distance $r = r(z, w)$ between points z and w in \mathbb{H} . Therefore, to ease the notation, we will denote the heat kernel $K_{\mathbb{H},k}(t; z, w)$ simply by $K_{\mathbb{H},k}(t, r)$.

In this section we extend Oshima’s result to all real values of k and prove the following theorem.

Theorem 3.1. *The heat kernel for the Maass-Laplacian D_k , for every $k \in \mathbb{R}$, is given by*

$$(3.1) \quad K_{\mathbb{H},k}(t, r) = \frac{e^{-\frac{t}{4}}}{(4\pi t)^{\frac{3}{2}}} \int_r^\infty \frac{ue^{-\frac{u^2}{4t}}}{\sqrt{2 \cosh u - 2 \cosh r}} \cdot \mathcal{T}_{2k} \left(\frac{\cosh \frac{u}{2}}{\cosh \frac{r}{2}} \right) du,$$

where

$$\mathcal{T}_{2k}(x) = \left(x + \sqrt{x^2 - 1}\right)^{2k} + \left(x - \sqrt{x^2 - 1}\right)^{2k}.$$

Proof. Our starting point is [18, formula (2.14)]

$$(3.2) \quad g(r) = -\frac{1}{\pi i} \int_{\text{Res}=\alpha>|k|} h(s) \left(s - \frac{1}{2}\right) Q_{s,k}(\cosh r) ds,$$

where $h(s)$ is a test function satisfying certain conditions and

$$Q_{s,k}(\cosh r) = -\frac{1}{4\pi} \int_r^{+\infty} e^{-(s-\frac{1}{2})u} \frac{I_k(u, r)}{\sqrt{2 \cosh u - 2 \cosh r}} du$$

is given by [5, formula (21)] with $n = 0$. Here

$$(3.3) \quad I_k(u, r) = (\cosh r - 1)^{-k} \cdot \left[e^{2k\Theta} \left(\sqrt{2} \sinh \frac{u}{2} - \sqrt{\cosh u - \cosh r} \right)^{2k} + e^{-2k\Theta} \left(\sqrt{2} \sinh \frac{u}{2} + \sqrt{\cosh u - \cosh r} \right)^{2k} \right],$$

where Θ satisfies the relation

$$(3.4) \quad e^{\pm\Theta} \sinh r = e^u - \cosh r \pm e^{\frac{u}{2}} \sqrt{2 \cosh u - 2 \cosh r},$$

see [5, p. 157].

Formula (3.2) is derived from the spectral decomposition theorem ([5, Theorem 1.5]) and holds true for any $k \in \mathbb{R}$. It is known that the heat kernel $K_{\mathbb{H},k}(t, r)$ is deduced by taking $h(s) = e^{s(s-1)t}$ in (3.2) and integrating along the vertical line $\text{Res} = \alpha$, for any $\alpha > |k|$. Therefore, application of Fubini-Tonelli theorem in (3.2) yields

$$(3.5) \quad K_{\mathbb{H},k}(t, r) = \frac{1}{4\pi^2 i} \int_r^{+\infty} H(u, t) \frac{I_k(u, r)}{\sqrt{2 \cosh u - 2 \cosh r}} du$$

where

$$H(u, t) = \int_{\text{Res}=\alpha>|k|} e^{s(s-1)t - (s-\frac{1}{2})u} \left(s - \frac{1}{2}\right) ds,$$

is actually independent of k . Moving the line of integration in the above integral to the line $\text{Res} = 1/2$, which is justified due to holomorphicity of the function under the integral sign and setting $s = 1/2 + iy$ we get

$$H(u, t) = -e^{-\frac{t}{4}} \int_{-\infty}^{+\infty} e^{-ty^2 - iuy} dy.$$

Applying [10, formula 3.462.6.] with $p = t > 0$ and $q = -iu/2$ we deduce that

$$H(u, t) = \frac{iu}{2t} \sqrt{\frac{\pi}{t}} e^{-\frac{t}{4}} e^{-\frac{u^2}{4t}}.$$

Inserting this into (3.5) we get

$$(3.6) \quad K_{\mathbb{H},k}(t, r) = \frac{e^{-\frac{t}{4}}}{(4\pi t)^{\frac{3}{2}}} \int_r^{\infty} \frac{ue^{-\frac{u^2}{4t}} I_k(u, r)}{\sqrt{2 \cosh u - 2 \cosh r}} du.$$

To complete the proof of the theorem it remains to prove that $I_k(u, r) = \mathcal{T}_{2k} \left(\frac{\cosh \frac{u}{2}}{\cosh \frac{r}{2}} \right)$.

Using (3.3), (3.4) and formula $\sinh^2 r = (\cosh r - 1)(\cosh r + 1)$ we get

(3.7)

$$I_k(u, r) = \frac{1}{(\cosh r - 1)^{2k} (\cosh r + 1)^k} \cdot \left[\left(e^u - \cosh r + e^{\frac{u}{2}} \sqrt{2 \cosh u - 2 \cosh r} \right)^{2k} \left(\sqrt{2} \sinh \frac{u}{2} - \sqrt{\cosh u - \cosh r} \right)^{2k} + \left(e^u - \cosh r - e^{\frac{u}{2}} \sqrt{2 \cosh u - 2 \cosh r} \right)^{2k} \left(\sqrt{2} \sinh \frac{u}{2} + \sqrt{\cosh u - \cosh r} \right)^{2k} \right].$$

Elementary transformations based on the properties of functions \sinh and \cosh yield that the first summand in the square bracket (3.7) is

(3.8)

$$\left[\left(e^u - \cosh r + e^{\frac{u}{2}} \sqrt{2 \cosh u - 2 \cosh r} \right) \cdot \left(\sqrt{2} \sinh \frac{u}{2} - \sqrt{\cosh u - \cosh r} \right) \right]^{2k} = (\cosh r - 1)^{2k} \left(\sqrt{2} \cosh \frac{u}{2} + \sqrt{\cosh u - \cosh r} \right)^{2k}.$$

Analogously, the second summand in the square bracket of (3.7) is
 (3.9)

$$\begin{aligned} & \left[\left(e^u - \cosh r - e^{\frac{u}{2}} \sqrt{2 \cosh u - 2 \cosh r} \right) \cdot \left(\sqrt{2} \sinh \frac{u}{2} + \sqrt{\cosh u - \cosh r} \right) \right]^{2k} \\ & = (\cosh r - 1)^{2k} \left(\sqrt{2} \cosh \frac{u}{2} - \sqrt{\cosh u - \cosh r} \right)^{2k}. \end{aligned}$$

Inserting (3.8) and (3.9) into (3.7), we get

$$\begin{aligned} I_k(u, r) = \frac{1}{(\cosh r + 1)^k} \cdot & \left[\left(\sqrt{2} \cosh \frac{u}{2} + \sqrt{\cosh u - \cosh r} \right)^{2k} \right. \\ & \left. + \left(\sqrt{2} \cosh \frac{u}{2} - \sqrt{\cosh u - \cosh r} \right)^{2k} \right]. \end{aligned}$$

Using that $\cosh r + 1 = 2 \cosh^2(r/2)$ we easily deduce the equality

$$I_k(u, r) = \mathcal{T}_{2k} \left(\frac{\cosh \frac{u}{2}}{\cosh \frac{r}{2}} \right).$$

The proof is complete. □

Remark 1. When $k \in \frac{1}{2}\mathbb{Z}$, the function $\mathcal{T}_{2k}(x)$ equals twice, the $2k$ th Chebyshev polynomial hence our result indeed generalizes the main result of [18].

Remark 2. The similar formula was derived by Alili, Matsumoto and Shiraiishi [1] by considering Schrödinger operator with constant magnetic field and using Gruet’s [11] calculation for Laplacian heat kernel.

4. PROPERTIES OF THE HEAT KERNEL

In this section we investigate properties of the heat kernel as a function of the hyperbolic distance r , as a function of the time $t > 0$ and as a function of the weight k .

4.1. Behavior of $K_{\mathbb{H},k}(t, r)$ as a function of r . Monotonicity of the heat kernel for the Maass-Laplacian, as a function of the hyperbolic distance r is proven by Friedman at al. in [7, Proposition 3.2], assuming that $k \in \frac{1}{2}\mathbb{Z}$. The following proposition generalizes that result to the case of all real weights k .

Proposition 4.1. *For any $t > 0$, the heat kernel $K_{\mathbb{H},k}(t, r)$ on \mathbb{H} associated to D_k , $k \in \mathbb{R}$, is strictly monotone decreasing as a function of $r > 0$.*

Proof. The proof follows the lines of the proof of [7, Proposition 3.2], so we will focus on parts that use the function \mathcal{T}_{2k} instead of the Chebyshev polynomial T_{2k} . We start with by writing the heat kernel $K_{\mathbb{H},k}(t, r)$ as

$$K_{\mathbb{H},k}(t, r) = \frac{e^{-\frac{t}{4}}}{\sqrt{2} (4\pi t)^{\frac{3}{2}}} \int_r^\infty \frac{\sinh u}{\sqrt{\cosh u - \cosh r}} F_k(t, r, u) du,$$

where

$$F_k(t, r, u) = \frac{ue^{-\frac{u^2}{4t}}}{\sinh u} \cdot \mathcal{T}_{2k} \left(\frac{\cosh \frac{u}{2}}{\cosh \frac{r}{2}} \right).$$

Integration by parts yields:

$$K_{\mathbb{H},k}(t, r) = -\frac{\sqrt{2}e^{-\frac{t}{4}}}{(4\pi t)^{\frac{3}{2}}} \int_r^\infty \frac{\partial}{\partial u} F_k(t, r, u) \sqrt{\cosh u - \cosh r} du.$$

In order to prove monotonicity of $K_{\mathbb{H},k}(t,r)$ we need to prove that $\frac{\partial}{\partial r}K_{\mathbb{H},k}(t,r) < 0$ for all $r > 0$. Proceeding analogously as in [7], we get:

$$(4.1) \quad \frac{\partial}{\partial r}K_{\mathbb{H},k}(t,r) = \frac{e^{-\frac{t}{4}}}{\sqrt{2}(4\pi t)^{\frac{3}{2}}} \int_r^\infty \left(\sinh u \frac{\partial}{\partial r}F_k(t,r,u) + \sinh r \frac{\partial}{\partial u}F_k(t,r,u) \right) \cdot \frac{du}{\sqrt{\cosh u - \cosh r}}.$$

Therefore, to prove the proposition it suffices to prove the inequality

$$(4.2) \quad \sinh u \frac{\partial}{\partial r}F_k(t,r,u) + \sinh r \frac{\partial}{\partial u}F_k(t,r,u) < 0, \quad \text{for all } r > 0.$$

It is trivial to deduce that

$$(4.3) \quad \begin{aligned} \sinh u \frac{\partial}{\partial r}F_k(t,r,u) + \sinh r \frac{\partial}{\partial u}F_k(t,r,u) &= \sinh r \left(\frac{1}{u} - \frac{u}{2t} - \frac{\cosh u}{\sinh u} \right) \cdot F_k(t,r,u) \\ &\quad + \mathcal{T}'_{2k}(x) \cdot \frac{ue^{-\frac{u^2}{4t}} \sinh \frac{r}{2}}{2 \cosh \frac{u}{2} \cosh^2 \frac{r}{2}} \left(\cosh^2 \frac{r}{2} - \cosh^2 \frac{u}{2} \right), \end{aligned}$$

where

$$(4.4) \quad \mathcal{T}'_{2k}(x) = \frac{2k}{\sqrt{x^2 - 1}} \left((x + \sqrt{x^2 - 1})^{2k} - (x - \sqrt{x^2 - 1})^{2k} \right)$$

and $x = \frac{\cosh \frac{u}{2}}{\cosh \frac{r}{2}}$.

For $u > 0$ it holds $\tanh u \leq u$, hence $\frac{\cosh u}{\sinh u} = \coth u \geq \frac{1}{u}$. Therefore,

$$\frac{1}{u} - \frac{u}{2t} - \frac{\cosh u}{\sinh u} \leq -\frac{u}{2t} < 0.$$

It is easy to check that $\mathcal{T}'_{2k}(x) > 0$ for all $x = \frac{\cosh \frac{u}{2}}{\cosh \frac{r}{2}} > 1$, where $\mathcal{T}'_{2k}(x)$ denotes the derivative with respect to x . Namely, when $k > 0$ both factors on the right-hand side of (4.4) are positive, while, for $k < 0$ both factors are negative.

Trivially, $\cosh^2 \frac{r}{2} - \cosh^2 \frac{u}{2} < 0$ for $u > r > 0$, since $\cosh^2 x$ is increasing function for $x > 0$. This proves that each term in the sum on the right-hand side of (4.3) is negative, hence (4.2) holds true.

The proof is complete. □

4.2. Behavior of $K_{\mathbb{H},k}(t,r)$ as a function of t . The small-time and large-time asymptotic behavior of the heat kernel is essential for its applications in physics, since the meromorphic continuation of the spectral zeta function associated to eigenvalues of the Maass-Laplacian relies on this information, see e.g. [16].

The following proposition describes the asymptotic behavior of $K_{\mathbb{H},k}(t,r)$ as $t \downarrow 0$ and $t \rightarrow \infty$.

Proposition 4.2. *The heat kernel $K_{\mathbb{H},k}(t,r)$ on \mathbb{H} associated to D_k , $k \in \mathbb{R}$, for all $r > 0$ satisfies the following two relations:*

$$(4.5) \quad K_{\mathbb{H},k}(t,r) = O\left(t^{-\frac{3}{2}}e^{-\frac{r^2}{4t}}\right), \quad \text{as } t \downarrow 0,$$

where the implied constant is independent of t , and

$$(4.6) \quad K_{\mathbb{H},k}(t,r) = O\left(e^{-\frac{t}{4}}t^{-\frac{3}{2}}\right), \quad \text{as } t \rightarrow \infty,$$

where the implied constant is independent of t .

Proof. We start by writing the heat kernel as

$$K_{\mathbb{H},k}(t,r) = \frac{e^{-\frac{t}{4}}e^{-\frac{r^2}{4t}}}{(4\pi t)^{\frac{3}{2}}} \int_r^\infty \frac{ue^{-\frac{u^2-r^2}{4t}}}{\sqrt{2 \cosh u - 2 \cosh r}} \cdot \mathcal{T}_{2k} \left(\frac{\cosh \frac{u}{2}}{\cosh \frac{r}{2}} \right) du.$$

The integral on the right-hand side of the above inequality can be written as the sum of two integrals $J_1(t,r)$, where the integration is along the interval $(r, r+1)$, and $J_2(t,r)$, where the integration is taken along the interval $(r+1, \infty)$. First, we will show that the integral $J_1(t,r)$ is finite and bounded by a constant independent of t . Namely, after substitution $x = \frac{\cosh \frac{u}{2}}{\cosh \frac{r}{2}}$, applying the inequality $e^{-\frac{u^2-r^2}{4t}} \leq 1$, we get

$$J_1(t,r) \leq \int_1^a \frac{\operatorname{arcosh}(x \cosh \frac{r}{2})}{\sqrt{x^2-1}} \mathcal{T}_{2k}(x) \frac{dx}{\sqrt{x^2 \cosh^2 \frac{r}{2} - 1}}$$

where $a = \cosh \frac{(r+1)}{2} \cosh^{-1} \frac{r}{2}$. Since $x \geq 1$ it obviously holds

$$J_1(t,r) \leq \frac{1}{\sqrt{2(\cosh^2 \frac{r}{2} - 1)}} \int_1^a \frac{\operatorname{arcosh}(x \cosh \frac{r}{2})}{\sqrt{x-1}} \mathcal{T}_{2k}(x) dx.$$

Taking into account that $\frac{\operatorname{arcosh}(x \cosh(r/2))}{\sqrt{x-1}} \mathcal{T}_{2k}(x) = O\left((x-1)^{-1/2}\right)$ as $x \searrow 1$ and $(x-1)^{-1/2}$ is integrable on $(1, a)$, we see that the integral on the right hand-side is finite. Hence, we have proved that $J_1(t,r)$ is bounded.

Next, we estimate $J_2(t,r)$. When $u > r+1$, there exists an absolute constant C , depending upon r , such that $(\cosh u - \cosh r)^{-1/2} \leq Ce^{-u/2}$. Since $\mathcal{T}_{2k}(x)$ increases for $x > 1$, from $\cosh(r/2) \geq 1$, we immediately deduce that

$$\mathcal{T}_{2k} \left(\frac{\cosh \frac{u}{2}}{\cosh \frac{r}{2}} \right) \leq \mathcal{T}_{2k} \left(\cosh \frac{u}{2} \right) \leq 2^{2|k|+2} e^{|k|u}.$$

Therefore,

$$(4.7) \quad J_2(t,r) \ll \int_{r+1}^\infty ue^{-\frac{u^2-r^2}{4t}} e^{(|k|-\frac{1}{2})u} du,$$

where the implied constant is independent of t . When $|k| - 1/2 < 0$, the above integral is obviously bounded by a constant which is independent of t , so it is left to estimate it in the case $|k| - 1/2 \geq 0$.

Until this point, we did not use any restrictions on t . Now, we will distinguish the cases $t \downarrow 0$ and $t \rightarrow \infty$.

First, assume that $t \downarrow 0$, then, we may assume $0 < t < (4(|k| - 1/2) + 4)^{-1}$, hence

$$J_2(t,r) \ll \int_{r+1}^\infty ue^{-(u^2-r^2)} e^{(|k|-\frac{1}{2})(r^2+\frac{1}{4}-(u-\frac{1}{2})^2)} du,$$

where the implied constant is independent of t . The integral on the right-hand side is obviously finite. Therefore, for $0 < t < (4(|k| - 1/2) + 4)^{-1}$ it yields that $J_2(t,r)$, and, hence, $J_1(t,r) + J_2(t,r)$ is bounded uniformly in t , which proves (4.5).

Next, we assume that t is large. Since the function under the integral sign on the right hand-side of (4.7) is non-negative, for $t > 1$ we have

$$J_2(t, r) \ll e^{\frac{r^2}{4t}} \int_0^\infty u e^{-\frac{u^2}{4t} + (|k| - \frac{1}{2})u} du.$$

The above integral can be explicitly computed as a function of t , $|k|$ and r using [10, formula 3.462.1.] with $\beta = 1/(4t) > 0$, $\gamma = -(|k| - 1/2)$ and $\nu = 2 > 0$ to get

$$(4.8) \quad J_2(t, r) \ll e^{\frac{r^2}{4t}} 2te^{\frac{t}{2}(|k| - \frac{1}{2})^2} D_{-2} \left(-\sqrt{2t} \left(|k| - \frac{1}{2} \right) \right),$$

where $D_{-\nu}(z)$ denotes the parabolic cylinder function (see, e.g. [10, p. 1037] for a definition). Using the asymptotic expansion of the parabolic cylinder function $D_{-2}(-\sqrt{2t}(|k| - 1/2))$, with $|z| = \sqrt{2t}(|k| - 1/2)$ large ([10, formula 9.246.1.]), we deduce that

$$D_{-2} \left(-\sqrt{2t} \left(|k| - \frac{1}{2} \right) \right) = \frac{1}{2t \left(|k| - \frac{1}{2} \right)^2} e^{-\frac{t}{2}(|k| - \frac{1}{2})^2} (1 + O(t^{-1})).$$

Inserting this into (4.8), we immediately deduce that for large enough t the integral $J_2(t, r)$ is bounded by a product of $e^{\frac{r^2}{4t}}$ and a function which is uniformly bounded in t . Since

$$K_{\mathbb{H},k}(t, r) = \frac{e^{-\frac{t}{4}}}{(4\pi t)^{\frac{3}{2}}} (J_1(t, r) + J_2(t, r)),$$

this proves (4.6).

The proof is complete. □

Remark 3. The asymptotic behavior (4.5) and (4.6) is important for various applications of the heat kernel. Namely, the upper half-plane is the universal cover for the Riemann surface \mathbb{H}/Γ , where Γ is a co-finite Fuchsian group of the first kind.

For example, (4.5) and (4.6) plus an additional consideration related to the small t asymptotics suffices to apply e.g. [13, Theorem 5.1] to $K_{\mathbb{H},k}(t, r)$.

In the case when the surface is compact and the multiplier system is well chosen (see e.g. [12, discussion on pp. 335-337]), the spectral expansion of the heat kernel, together with (4.5) and (4.6) are sufficient to deduce the meromorphic continuation of the spectral zeta function, following the approach described in [22].

We leave those applications to a subsequent paper.

4.3. Behavior of $K_{\mathbb{H},k}(t, r)$ as a function of k . Finally, Theorem 3.1 allows us to investigate monotonicity of the heat kernel for the Maass-Laplacian as a function of the weight k .

Proposition 4.3. *For any $t > 0$ and $r > 0$, the heat kernel $K_{\mathbb{H},k}(t, r)$, as a function of k , is strictly monotone decreasing for $k < 0$ and strictly monotone increasing for $k > 0$.*

Proof. First let us show that $K_{\mathbb{H},k}(t, r)$ is differentiable as a function of a real parameter k . The function under the integral sign in (3.1) is differentiable as a function of k and its derivative with respect to k equals

$$\frac{2ue^{-\frac{u^2}{4t}}}{\sqrt{2 \cosh u} - 2 \cosh r} \log \left(x + \sqrt{x^2 - 1} \right) \left[\left(x + \sqrt{x^2 - 1} \right)^{2k} - \left(x - \sqrt{x^2 - 1} \right)^{2k} \right],$$

where $x = \frac{\cosh \frac{u}{2}}{\cosh \frac{r}{2}}$. Therefore, to prove differentiability of $K_{\mathbb{H},k}(t, r)$ it is left to prove that, uniformly in k , for k in some interval, say $k \in (-M, M)$, $M > 0$, this function is bounded by an integrable function on (r, ∞) .

First, it is obvious that, for a fixed (large) $R > 0$, the function

$$\frac{ue^{-\frac{u^2}{4t}}}{\sqrt{2 \cosh u - 2 \cosh r}} \cdot \frac{d}{dk} \left(\mathcal{T}_{2k} \left(\frac{\cosh \frac{u}{2}}{\cosh \frac{r}{2}} \right) \right) \chi_{(r,R)}(u),$$

where $\chi_{(r,R)}(u)$ denotes the characteristic function of the interval (r, R) , is bounded by an integrable function, uniformly in $k \in (-M, M)$. For large enough R , for all $u \geq R$ and $k \in (-M, M)$ one has

$$\left| \frac{d}{dk} \mathcal{T}_{2k} \left(\frac{\cosh \frac{u}{2}}{\cosh \frac{r}{2}} \right) \right| \leq Aue^{(M+\frac{1}{2})u},$$

where A is a positive constant independent of k . The last inequality yields that

$$\left| \frac{ue^{-\frac{u^2}{4t}}}{\sqrt{2 \cosh u - 2 \cosh r}} \cdot \frac{d}{dk} \mathcal{T}_{2k} \left(\frac{\cosh \frac{u}{2}}{\cosh \frac{r}{2}} \right) \chi_{[R,\infty)}(u) \right| \leq Aue^{-\frac{u^2}{4t} + (M+\frac{1}{2})u}.$$

By [10, formula 3.462.1.] with $\beta = 1/(4t) > 0$, $\gamma = -(M + 1/2)$ and $\nu = 2 > 0$ we see that the function $ue^{-\frac{u^2}{4t} + (M+\frac{1}{2})u}$ is integrable on $(0, \infty)$ and hence, by positivity, it is integrable on $[R, \infty)$. Therefore, we can conclude that for $k \in (-M, M)$, $M > 0$, the derivative

$$\frac{2ue^{-\frac{u^2}{4t}}}{\sqrt{2 \cosh u - 2 \cosh r}} \log \left(x + \sqrt{x^2 - 1} \right) \left[\left(x + \sqrt{x^2 - 1} \right)^{2k} - \left(x - \sqrt{x^2 - 1} \right)^{2k} \right]$$

is bounded by an integrable function on (r, ∞) , uniformly in $k \in (-M, M)$. Since $M > 0$ is arbitrary, by the theorem on differentiation under the integral sign it follows that $K_{\mathbb{H},k}(t, r)$ is a differentiable as a function of a real parameter k and

$$\frac{d}{dk} K_{\mathbb{H},k}(t, r) = \int_r^\infty \frac{ue^{-\frac{u^2}{4t}}}{\sqrt{2 \cosh u - 2 \cosh r}} \cdot \frac{d}{dk} \left(\mathcal{T}_{2k} \left(\frac{\cosh \frac{u}{2}}{\cosh \frac{r}{2}} \right) \right) du.$$

By writing $\mathcal{T}_{2k}(x) = 2 \cosh(2k \log(x + \sqrt{x^2 - 1}))$, one immediately gets $\frac{d}{dk} \mathcal{T}_{2k}(x) < 0$ for all $k < 0$ and $\frac{d}{dk} \mathcal{T}_{2k}(x) > 0$ for all $k > 0$. Namely,

$$\frac{d}{dk} \mathcal{T}_{2k}(x) = 4 \log \left(x + \sqrt{x^2 - 1} \right) \sinh \left(2k \log \left(x + \sqrt{x^2 - 1} \right) \right)$$

which is of the same sign as k since $x = \frac{\cosh \frac{u}{2}}{\cosh \frac{r}{2}} > 1$ and the function \sinh is an odd function. Therefore, we conclude that for arbitrary, but fixed $t > 0$ and $r > 0$ we have $\frac{d}{dk} K_{\mathbb{H},k}(t, r) < 0$ for $k < 0$ and $\frac{d}{dk} K_{\mathbb{H},k}(t, r) > 0$ for $k > 0$.

The proof is complete. □

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REFERENCES

- [1] L. Alili, H. Matsumoto, T. Shiraishi, On a triplet of exponential Brownian functionals, In: J. Azéma, M. Émery, M. Ledoux, M. Yor (eds), Séminaire de Probabilités XXXV. Lecture Notes in Mathematics 1755, Springer, Berlin, Heidelberg, 2001, pp. 396–415. https://doi.org/10.1007/978-3-540-44671-2_27.
- [2] C. Burrin, Generalized Dedekind sums and equidistribution mod 1, *J. Number Theory* 172 (2017) 270–286. <https://doi.org/10.1016/j.jnt.2016.08.003>.
- [3] C. Burrin, J. Jorgenson, C. O’Sullivan, L. Smajlović, Transformation laws for generalized Dedekind sums associated to Fuchsian groups, arXiv:1904.11333 [math.NT].
- [4] E. D’Hoker, D.H. Phong, The geometry of string perturbation theory, *Rev. Mod. Phys.* 60 (1988) 917–1065. <https://doi.org/10.1103/RevModPhys.60.917>.
- [5] J.D. Fay, Fourier coefficients of the resolvent for a Fuchsian group, *J. Reine Angew. Math.* 293/294 (1977) 143–203. <http://eudml.org/doc/151896>.
- [6] J. Fischer, An approach to the Selberg trace formula via the Selberg zeta-function, *Lecture Notes in Mathematics* 1253, Springer-Verlag, Berlin - Heidelberg, 1987.
- [7] J.S. Friedman, J. Jorgenson, J. Kramer, Uniform sup-norm bounds on average for cusp forms of higher weights, *Arbeitstagung Bonn 2013* (2016) 123–154. https://link.springer.com/chapter/10.1007/978-3-319-43648-7_6.
- [8] J.S. Friedman, J. Jorgenson, J. Kramer, Effective sup-norm bounds on average for cusp forms of even weight, *Trans. Amer. Math. Soc.* 372 (2019) 7735–7766. <https://doi.org/10.1090/tran/7933>.
- [9] J.S. Friedman, J. Jorgenson, L. Smajlović, The determinant of the Lax-Phillips scattering operator, arXiv:1603.07613 [math.NT].
- [10] I.S. Gradshteyn, I.M. Ryzhik, *Table of integrals, series and products*, eighth ed., Elsevier Academic Press, 2014. <https://doi.org/10.1016/C2010-0-64839-5>.
- [11] J.C. Gruet, Semi-groupe du mouvement brownien hyperbolique, *Stochastics and Stochastics Reports* 56 (1996) 53–61. <https://doi.org/10.1080/17442509608834035>.
- [12] D.A. Hejhal, *The Selberg Trace Formula for $PSL(2, \mathbf{R})$* , vol. 2, *Lecture Notes in Mathematics* 1001, Springer-Verlag, New York, 1983.
- [13] J. Jorgenson, S. Lang, Analytic continuation and identities involving heat, Poisson, wave and Bessel kernels, *Math. Nachr.* 258 (2003) 44–70. <https://doi.org/10.1002/mana.200310086>.
- [14] J. Jorgenson, C. O’Sullivan, L. Smajlović, Modular Dedekind symbols associated to Fuchsian groups and higher-order Eisenstein series, arXiv:1610.06841 [math.NT].
- [15] H. Maass, Die Differentialgleichungen in der Theorie der elliptischen Modulfunktionen, *Math. Ann.* 125 (1952) 235–263. <https://doi.org/10.1007/BF01343120>.
- [16] W. Müller, Spectral theory for Riemannian manifolds with cusps and a related trace formula, *Math. Nachr.* 111 (1983) 197–288. <https://doi.org/10.1002/mana.19831110109>.
- [17] P. Lax, R. Phillips, *Scattering theory for automorphic functions*, *Annals of Mathematics Studies* No. 87, Princeton University Press, Princeton, New Jersey, 1976.
- [18] K. Oshima, Completeness relations for Maass Laplacians and heat kernels on the super Poincaré upper half-plane, *J. Math. Phys.* 31 (1990) 3060–3063. <https://doi.org/10.1063/1.528959>.
- [19] A.M. Polyakov, Quantum geometry of bosonic strings, *Phys. Lett.* 103B (1981) 207–210. [https://doi.org/10.1016/0370-2693\(81\)90743-7](https://doi.org/10.1016/0370-2693(81)90743-7).
- [20] D.B. Ray, I.M. Singer, R-torsion and the Laplacian on Riemannian manifolds, *Advances in Math.* 7 (1971) 145–210. [https://doi.org/10.1016/0001-8708\(71\)90045-4](https://doi.org/10.1016/0001-8708(71)90045-4).
- [21] W. Roelcke, Analytische Fortsetzung der Eisensteinreihen zu den parabolischen Spitzen von Grenzkreisgruppen erster Art, *Math. Ann.* 132 (1956/57) 121–129. <http://eudml.org/doc/160508>.
- [22] P. Sarnak, Determinants of Laplacians, *Comm. Math. Phys.* 110 (1987) 113–120. <https://projecteuclid.org/euclid.cmp/1104159171>.