ALBANIAN JOURNAL OF MATHEMATICS
Volume 13, Number 1, Pages 201–210
ISSN: 1930-1235; (**2019**)

SPIN STRUCTURES, THETA FUNCTIONS AND TOPOLOGICAL INSULATORS

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Abstract. We investigate a relationship between topological insulators and spin structures on Riemann surfaces, along with their theta function interpretation, via Quillen's determinant bundle. The genus one case is analyzed in detail and it is found to yield a non trivial Kane-Mele type invariant.

MSC 2010: 53D50, 81S10. 14H42, 14H40.
Keywords: Topological insulators, Spin^c and Spin structures, theta functions, geometric quantization.

1. Introduction

Topological insulators, also called quantum spin Hall systems, represent a major topic in modern condensed matter physics. They are materials which behave as insulators in their bulk, i.e. they admit an energy gap separating the valence and conduction bands, whilst exhibiting gapless topologically protected conducting edge or surface states on their boundary (surviving impurities or geometric perturbations), with the topological protection arising from spin-orbit coupling together with time-reversal or other symmetries of the underlying one-particle Hamiltonians (see e.g. [17] for a recent in depth review and also the agile survey [28]). As such, they fostered fascinating mathematical problems requiring, for their solution, extensive use of sophisticated techniques in geometry and topology, notably classical and noncommutative geometry and K-theory ([12, 13, 20, 21, 22, 27, 23]).

In this short note we discuss a quite simple two-dimensional toy model exhibiting such a behaviour, by resorting to spin geometry in connection with theta characteristic theory, building on the classical papers [2, 25], see also [26]. A crucial tool

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is provided by Quillen's theorem [29], in a form used in early string theory investigations by Alvarez-Gaumé *et al.* ([1]). In a nutshell, we consider the spinor bundle over a complex 2-torus, identified with its Jacobian and with its Picard group, together with a family of Spin^c operators thereon, taken as a family of Landau-Bloch Hamiltonians. The common spectrum (band structure) of the family is easily determined and it is then shown that enforcement of time-reversal invariance of the Hamiltonian selects the points corresponding to the four spin structures on the torus. The ensuing harmonic spinors, i.e. the zero modes of the Dirac operator (the eigenvalue zero is the corresponding Fermi energy) exist only in the odd spin structure case and play the role of the topologically protected edge states. Then, Quillen determinant ideas lead via [1] to the theta characteristics, coming full circle with the portraits of Atiyah and Mumford ([2, 25]). Everything is extended to higher genus, which, however, appears to be topologically less interesting in the sense that in this case one arrives at a trivial Kane-Mele type invariant, in contrast to the genus one case. Nevertheless, spin theoretic ideas could be relevant in investigating possible metric (actually complex structure) dependent physical phenomena. At the same time, a topological insulator perspective may cast further light on a celebrated classical theory.

The paper is organized as follows. In Section 2 we briefly review basic facts and terminology on Bloch bundle theory, together with spin structures on Riemann surfaces and their relationship to theta function theory, in view of the formulation of the result presented in Section 3 and extended to higher genus in Section 4. The final section summarizes the discussion and points out further research directions. The two appendices aim at adding a few details to the main text discussion and in particular at recording the explicit eigenspinors in the torus case.

2. Preliminaries

In this section, split into several smaller ones, we collect miscellaneous mathematical and physical background, referring to the cited literature for full information, and at the same time we develop the necessary calculations leading to the main result.

2.1. **Bloch bundles.** In the present section Bloch bundles associated to time-reversal ($\mathcal{T}$) invariant spin 1/2-systems are considered (see e.g. [17, 20, 21, 22, 12, 13, 24, 11]). In the literature, they are known as topological insulators in class AII. These are vector bundles

$$\mathfrak{B} \to \mathcal{B}$$

on the so-called Brillouin (or spectral) manifold $\mathcal{B}$ (yielding the admissible wave vectors $\mathbf{k}$) whose fibres are eigenspaces of Bloch Hamiltonians. The Kramers degeneracy theorem states that the energy levels in such a system are doubly degenerate, i.e. the (finite dimensional) eigenspaces are even-dimensional. The $\mathcal{T}$-action on $\mathcal{B}$ (concisely reading $\mathbf{k} \mapsto -\mathbf{k}$) prompts the existence of an antiunitary operator Θ on sections of $\mathfrak{B}$ such that $\Theta^2 = -I$, leading to a topological isomorphism $\mathfrak{B} \cong \overline{\mathfrak{B}}$ (the conjugate bundle) and thus to the Chern class identities

$$c_i(\mathfrak{B}) = (-1)^i c_i(\mathfrak{B}), \quad i = 1, 2 \dots$$

If the base manifold is two-dimensional, we just have

$$c_1(\mathfrak{B}) = c_1(\det \mathfrak{B}) = 0,$$

that is, the determinant line bundle $\det \mathfrak{B}$ is topologically trivial. The fixed points of the $\mathcal{T}$-action on $\mathcal{B}$ play a special role. In Section 2.3 below we are going to exhibit an example based on Riemann surface theory explicitly featuring the typical characteristics of topological insulators.

2.2. **Dirac operators and harmonic spinors on Riemann surfaces.** In this section we succinctly collect, in order to establish notation and for the reader's benefit, well known facts concerning the spin geometry of a compact Riemann surface Σ_g of genus $g \geq 1$ (see e.g. [16, 4, 2, 18, 10] for background).

Let $E \to \Sigma_g$ be a holomorphic vector bundle over Σ_g, with rank $\mathrm{rk}(E)$ and degree $\deg(E)$, and let $\overline{\partial}_E$ be the $\overline{\partial}$-operator attached to it, with $h^0(E)$ being the dimension of the space of its holomorphic sections. Let $K \to \Sigma_g$ be the canonical bundle, having degree $\deg K = -\chi(\Sigma_g) = 2g-2$ and $h^0(K) = g$, its holomorphic sections being the Abelian differentials. Then the Riemann-Roch theorem reads (combined with Serre duality)

$$\mathrm{ind}(\overline{\partial}_E) = h^0(E) - h^1(E) = h^0(E) - h^0(K \otimes E^*) = \deg(E) + \mathrm{rk}(E)(1-g)$$

($E^* \to \Sigma_g$ is the dual bundle). A crucial fact is now that the *spin structures* on Σ_g precisely correspond to the possible (indeed, 2^{2g}) *holomorphic square roots* $K^{\frac{1}{2}}$ of the canonical bundle. The (Spin^c) Dirac operator $\mathcal{D}_E$ pertaining to E coincides, after twisting with a holomorphic square root of K, with $\sqrt{2}(\overline{\partial}_E + \overline{\partial}_E^{\dagger})$. The chiral Dirac operator D^+ is then $\overline{\partial}_E$. The full Dirac operator is (up to a constant)

$$\mathcal{D}_E = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix},$$

with $D^- = (D^+)^{\dagger}$, the (formal) adjoint of D^+. An application of Riemann-Roch to $E = L = K^{\frac{1}{2}}$ yields, since $\deg(L) = g-1$ and $\mathrm{ind}(\overline{\partial}_L) = 0$, $h^0(L) \equiv h^0 = h^1(L) \equiv h^1$.

The space H of *harmonic spinors*, consisting of the solutions of the Dirac equation $\mathcal{D}_L\psi \equiv \mathcal{D}\psi = 0$ decomposes as $H = H^+ \oplus H^-$ (positive and negative harmonic spinors) and has dimension

$$h = h^+ + h^- = h^0 + h^1 = 2h^0.$$

In general, a spin structure α is called even (resp. odd) if its corresponding $h^0_\alpha = \dim H^0(L_\alpha)$ is even (respectively odd): in our case $2^{g-1}(2^g \pm 1)$ of them are even resp. odd. We denote by $\mathfrak{A}$ the set of all spin structures and by $\mathfrak{A}^{\pm}$ the set of the even and odd ones, respectively. In the torus case Σ_1 the canonical bundle K is topologically trivial and one has four spin structures, of which three are even and one odd, with the properties $h = h^+ + h^- = 2h^+ = 2$ for the single *odd* spin structure, and $h = 0$ for the *even* ones. The bundles L_α, $\alpha \in \mathfrak{A}$ are, of course, topologically trivial as well (therefore, in physical parlance, no magnetic field is present); see [18, 10] for complete details.

2.3. **The torus case.** This subsection is devoted to a quite explicit discussion of the genus one case, where more clear-cut and, in a sense to be made precise below, more interesting results can be obtained. Take as $\mathfrak{B} \to \mathcal{B}$ the trivial vector bundle over the Jacobian $J(\Sigma_1)$ - identified with Σ_1 and with $\mathrm{Pic}_0(\Sigma_1)$, the group of degree zero holomorphic line bundles - whose fibre at each point is the full spinor space, identified in turn with $L^2(\Sigma_1, \mathbb{C}^2)$ - Σ_1 being equipped with the standard area form

- with the following family of Spin^c operators $\mathcal{D}_a$ (Landau-Bloch Hamiltonians) acting thereon:

$$\mathcal{D}_a = \begin{bmatrix} 0 & D_a^- \\ D_a^+ & 0 \end{bmatrix},$$

with $a = a_1 + ia_2 \in J(\Sigma_1) = \mathbb{C}/\Lambda \cong \Sigma_1$ (Λ being the Jacobian lattice generated, for simplicity, by 1 and i), parametrized, for instance, by the fundamental region $[0, 2\pi) \times [0, 2\pi)$. The operators $D_a^\pm$ read, explicitly:

$$D_a^+ = \frac{\partial}{\partial \overline{z}} + a, \qquad D_a^- = (D_a^+)^\dagger = -\frac{\partial}{\partial z} + \overline{a},$$

the same (periodic) boundary conditions being employed for all operators. In particular, for $a = 0$, $D_0^\pm \equiv D^\pm$, or $\mathcal{D}_0 \equiv \mathcal{D}$. This operator corresponds to the only *odd* spin structure (the (P, P) spin structure in the notation of [1]). The *symmetric points* on $J(\Sigma_1)$, namely those z satisfying

$$z = -z \bmod \Lambda \iff 2z \in \Lambda$$

provide the four $\mathcal{T}$-invariant points in the Brillouin manifold $J(\Sigma_1) = \Sigma_1 = \mathrm{Pic}_0(\Sigma_1)$ and correspond, in turn, to the four spin structures. The latter will be selected upon enforcement of time-reversal invariance within the chosen family of Hamiltonians.

Upon denoting a generic spinor (later to be taken in the domain of the operators involved) by

$$\psi = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

one can write down the standard anti-unitary *time-reversal operator* Θ, satisfying $\Theta^2 = -I$, reading

$$\Theta\,\psi = i\sigma_y \begin{bmatrix} \overline{\xi} \\ \overline{\eta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \overline{\xi} \\ \overline{\eta} \end{bmatrix} = \begin{bmatrix} \overline{\eta} \\ -\overline{\xi} \end{bmatrix}.$$

A straightforward computation shows that the $\mathcal{T}$-invariance property

$$\mathcal{D}_a \circ \Theta = \Theta \circ \mathcal{D}_a$$

holds if and only if

$$2a \in \Lambda,$$

namely, for a fixed point for the $\mathcal{T}$-action and, correspondingly, for a genuine spin structure on Σ_1. Indeed, we have

$$\overline{D_a^+ \xi} = -D_{-a}^- \overline{\xi}, \qquad \overline{D_a^- \eta} = -D_{-a}^+ \overline{\eta}.$$

Therefore, on the one hand

$$\Theta\,\mathcal{D}_a \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \overline{D_a^+ \xi} \\ -\overline{D_a^- \eta} \end{bmatrix}$$

and, on the other hand:

$$\mathcal{D}_a\,\Theta \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} -D_a^- \overline{\xi} \\ D_a^+ \overline{\eta} \end{bmatrix} = \begin{bmatrix} \overline{D_{-a}^+ \xi} \\ -\overline{D_{-a}^- \eta} \end{bmatrix},$$

whence our conclusion follows. This immediately yields *Kramers' degeneracy*: if ψ is an eigenfunction of $\mathcal{D}_\alpha$ (the Dirac operator associated to the spin structure $\alpha \in \mathfrak{A}$) corresponding to an eigenvalue ε, so is $\Theta\psi$, which is orthogonal to ψ.

2.4. **Band structure.** Pursuing the computation (see Appendix B for more details) one finds, from

$$D_a^+ D_a^- = D_a^- D_a^+ = -\frac{\partial^2}{\partial z \partial \bar{z}} + \bar{a}\frac{\partial}{\partial \bar{z}} - a\frac{\partial}{\partial z} + \bar{a}a = -\frac{\Delta}{4} + i(a_1\frac{\partial}{\partial y} - a_2\frac{\partial}{\partial x}) + a_1^2 + a_2^2,$$

the following spectrum of $\mathcal{D}_a^2$ (*band structure*):

$$[\varepsilon_{m_1 m_2}(a)]^2 = \frac{m_1^2 + m_2^2}{4} + m_1 a_2 - m_2 a_1 + a_1^2 + a_2^2 = \| \frac{\mathbf{m}}{2} - \tilde{\mathbf{a}} \|^2 \qquad m_1, m_2 \in \mathbb{Z}$$

(with $\mathbf{m} = (m_1, m_2)$ and $\tilde{\mathbf{a}} = (a_2, -a_1)$), also cf. [33]. Therefore $\varepsilon_{m_1 m_2}(a) = \varepsilon_F = 0$ ("Fermi energy") if and only if $a_1 = a_2 = m_1 = m_2 = 0$. This vividly confirms the absence of harmonic spinors for the even spin structures and the emergence of a *Dirac cone* structure for $\mathcal{D}_0 = \mathcal{D}$ (signalling a crossing of the Fermi energy $\varepsilon_F = 0$). Indeed, for $\mathbf{m} = 0$, one has

$$[\varepsilon_{00}(a)]^2 = \| \tilde{\mathbf{a}} \|^2 = a_1^2 + a_2^2$$

and we have two natural (actually artificial!) "conduction" and "valence" families of bands given by $a \mapsto \pm|\varepsilon_{m_1 m_2}(a)|$, respectively. The bands $a \mapsto \pm|\varepsilon_{00}(a)| = \pm\sqrt{a_1^2 + a_2^2}$ *intersect* at $a = 0$.

In general, *the parity of the number of Dirac cones* (coinciding with the pairs of *Majorana zero modes*) yields (a version of) the so-called *Kane-Mele invariant* KM ([19, 17, 20, 21, 22, 12, 13, 23]) which is then $\mathbb{Z}_2$-valued. A topological insulator has $KM = -1$. This is exactly what happens here: the Majorana zero modes are the harmonic spinors. This point will be further elaborated on in Section 3.

Remark. The previous calculation shows the non trivial Pic_0-dependence of the spectrum of D^+, typical of the degree zero case. In fact, for a positive degree (holomorphic) line bundle (thus possessing non trivial holomorphic sections) one would have isospectrality (see [33] and also [30]).

2.5. **Determinant line bundles on Riemann surfaces.** Dan Quillen introduced, in [29], the *determinant line bundle* associated to a family of Cauchy-Riemann operators D on a Riemann surface Σ and developed its geometry. Referring to that paper or to [14], and also to [1] for complete details, we just recall here that the Ray-Singer ζ-regularized determinant of the family admits the following representation:

$$\det{}_\zeta(D^\dagger D) = e^{-q(D)} |\det(D; D_0)|^2,$$

where

$$q(D) = \| D - D_0 \|^2 = \frac{i}{2\pi}\int_\Sigma \mathrm{tr}_E[(D - D_0)^\dagger (D - D_0)],$$

D_0 being a fixed operator in the family, and $\det(D; D_0)$ is a *holomorphic* section vanishing precisely when a specific operator D has a non trivial kernel. Quillen's result has been vastly generalized (see e.g. [5, 6, 7, 8, 9]).

2.6. **Theta functions.** Recall that any holomorphic line bundle over the torus Σ_1 is a multiple of the so-called *theta line bundle*, possessing - by Riemann-Roch - a single holomorphic section (up to a scalar), pulling back to the well-known theta function on $\mathbb{C}$. The *theta functions with real characteristics* $\hat{a}$, $\hat{b}$, in the $g = 1$ case read

$$\vartheta\begin{bmatrix}\hat{a}\\ \hat{b}\end{bmatrix}(z, \tau) = \sum_{n\in\mathbb{Z}} \exp\left[\pi i(n + \hat{a})\,\tau(n + \hat{a}) + 2\pi i(n + \hat{a})(z + \hat{b})\right]$$

For $\hat{a} = \hat{b} = 0$ one gets the standard theta function ϑ. In the present paper we take $\tau = i$, see [26, 16, 1] for complete details.

3. The main result (genus one case)

Application of the above considerations to the family $\mathcal{D}^+ := \{D_a^+\}$, $a \in J(\Sigma_1)$ shows that $\det_\zeta((\mathcal{D}^+)^\dagger \mathcal{D}^+) = 0$ precisely for $a = 0$, the zero is simple and the kernel is 1-dimensional. Consequently, taking $D_0 = D_0^+$, the section $\det(D; D_0)$ is proportional to a *theta* section: specifically, the theta-null formula

$$\vartheta \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} (0|i) = e^{-\pi \hat{a}^2} e^{2\pi i \, \hat{a}\hat{b}} \, \vartheta(i\hat{a} + \hat{b}, i) \equiv c(\hat{a}, \hat{b}) \vartheta(u, i)$$

shows that, *up to the elementary factor* $c(\hat{a}, \hat{b})$, the l.h.s. is a standard theta function, so it depends holomorphically on $u := i\hat{b} + \hat{a}$, and this matches with $\deg(\det \mathcal{D}^+) = 1$. In particular $\det(D_0; D_0) = 0$ corresponds to $\vartheta\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right](0|i) = \vartheta(1/2 + (1/2)i, i) = 0$, see also [1].

Also notice that

$$\det \mathcal{D} = \det \mathcal{D}^+ \otimes \det \mathcal{D}^- = 1 \qquad (\diamondsuit)$$

(i.e. the trivial line bundle, cf. Section 2.1).

Let us now relate the above considerations to the original approach of Kane-Mele [19] together with its revisitation and enhancement in the papers [20, 21, 22]. The determinant line bundle $\det \mathcal{D}^+ =: \operatorname{Pf} \mathcal{D}$ may be read as a *holomorphic* Pfaffian, in the sense that $(\diamondsuit)$ becomes the factorization

$$\det \mathcal{D} = \operatorname{Pf} \mathcal{D} \otimes (\operatorname{Pf} \mathcal{D})^*.$$

The fibre $\det \mathcal{D}|_0$ at the odd spin structure (corresponding to $a = 0$) is generated by $e^+ \wedge \Theta e^+ = -e^+ \wedge e^-$ in the notation of Appendix A. As a genuine *square root* thereof, denoted as $\operatorname{Pf} \mathcal{D}|_0 \cong (\operatorname{Pf} \mathcal{D}|_0)^*$, we can take the line generated by e^+ or, equivalently, the one generated by $\Theta e^+$and we get the Majorana zero-mode portrait hinted at above.

The full discussion may now be summarized via the following

Theorem 3.1. *The above Bloch bundle features a single "Dirac cone", corresponding to the unique* odd *spin structure, signalled by* $\vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (0|i) = 0$, *thus yielding a* topological insulator*, with a two-dimensional space of harmonic spinors (a manifestation of Kramers degeneracy).*

Remark. A possible *spin version* of the $(\mathbb{Z}_2)$ Kane-Mele invariant ([19, 17]) would read

$$KM_{\text{spin}} := \prod_{\alpha \in \mathfrak{S}} (-1)^{h_\alpha^0} = \prod_{\alpha \in \mathfrak{S}^-} (-1)^{h_\alpha^0}$$

Thus the above result yields, for our system, the *non trivial* outcome

$$KM_{\text{spin}} = -1.$$

4. Extension to higher genus

Since the line bundles $L = L_\alpha$ are now topologically non trivial ($\deg(L_\alpha) = g-1$), we have to distinguish L from its dual L^{-1}. The complex conjugation is to be replaced by the antilinear map (cf. [2])

$$\hat{h} : \xi \mapsto h\overline{\xi}$$

(h being a hermitian metric on L) sending sections of L to sections of L^{-1}. The adjoint of the $\overline{\partial}$-operator pertaining to an initial spin structure, when acting on sections of L^{-1}, then reads locally

$$\overline{\partial}^\dagger \equiv D^- = -\frac{\partial}{\partial z} + \frac{\partial \log h}{\partial z}$$

whence

$$\hat{h} \circ D^+ = -D^- \circ \hat{h}$$

and the same holds for the extension to $\{\mathcal{D}_a\}$, reading, explicitly, and locally (obvious notation, as in [16])

$$D_a^+ = \frac{\partial}{\partial \bar{z}} + \sum_{i=1}^{g} a^i \overline{\left[\frac{\omega_i}{dz}\right]} \equiv \frac{\partial}{\partial \bar{z}} + a \cdot \overline{\left[\frac{\omega}{dz}\right]}$$

where the $\{\omega_i\}$ are a basis of *Abelian differentials* and a belongs to the Jacobian $J(\Sigma_g) = \mathbb{C}^g/\Lambda$:

$$\hat{h} \circ D_a^\pm = -D_{-a}^\mp \circ \hat{h}.$$

Therefore, upon defining the time-reversal operator Θ via

$$\Theta = i\sigma_y \circ \hat{h} = \hat{h} \circ i\sigma_y,$$

enforcement of the condition

$$\Theta \circ \mathcal{D}_a = \mathcal{D}_a \circ \Theta$$

again leads to $2a = 0$ in $J(\Sigma_g)$, selecting the 2^{2g} *bona fide* spin structures. We explicitly notice that, in the latter situation, the operator $\hat{h} \circ \mathcal{D}_a^+ = -\mathcal{D}_{-a}^- \circ \hat{h}$ is essentially Atiyah's operator P of [2], with $\mathbb{Z}_2$-index equal to $(-1)^{h_\alpha^0}$ (α being the spin structure corresponding to a), see [3, 2]. Also cf. [20, 21, 22] for a related Atiyah-Singer interpretation of the KM invariant.

Thus, proceeding exactly as before (in particular, following [1, 26] for the connection with theta function theory) we would obtain in this case, for the hypothetical KM-spin invariant detecting the *parity* of the number of Dirac cones introduced in the previous section

$$KM_{\text{spin}} = 1.$$

Indeed for $g \geq 2$ there is an even number of odd spin structures, while h_α^0 is odd, whence, *from a strictly topological point of view, the net result is trivial in higher genus*. Nevertheless, the fact that, in general the space of harmonic spinors *varies with the metric* ([18, 1, 10]) hints at the possibility of physical models reflecting this characteristic. In particular, the quite different behaviours of hyperelliptic and non hyperelliptic Riemann surfaces (the latter existing for $g \geq 3$) might explicitly show up. Also, the band structure could be described upon suitably adapting the methods employed in [33] to our specific case. However, it is a priori clear that the spectrum varies, since already h_α^0 varies with the spin structure α.

5. Concluding remarks and outlook

The structure described in this work is in a sense archetypical, in that it possibly exhibits the simplest model of topological insulator, albeit a purely mathematical one. The presence of the Picard variety (Brillouin manifold), also parametrizing flat connections modulo gauge equivalence, clearly shows that, in genus one, in contrast to the Quantum Hall Effect, no magnetic field (curvature) is needed (the latter would indeed break time-reversal invariance). The extension to Riemann surfaces of genus $g \geq 2$ apparently does not add further strictly topological insights, as we have seen. Nevertheless, experimentalists may possibly devise materials exhibiting the above (in principle unstable for higher genus) Dirac cone structures. However, further elaboration of harmonic spinor ideas may be useful for investigation of higher dimensional topological insulators as well and for additional, metric (or, better, complex structure) dependent, exotic properties of matter. At the same time, we have offered a topological insulator reinterpretation of a small part of a venerable classical theory, which might deserve further investigation.

Appendix A

Here we briefly review the simplest instance of the Fock representation of the CAR and Clifford algebras, closely following [32, 15], with a few minor modifications, in order to complement the discussion given in the main text.

Let $F = \mathbb{C}^2$, with its canonical basis (e^+, e^-):

$$e^+ := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad e^- := \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Also set $F^\pm = \langle e^\pm \rangle$. Then one has

$$F = \mathbb{C}^2 \cong \Lambda^*(\mathbb{C}) = \Lambda^0\mathbb{C} \oplus \Lambda^1\mathbb{C} = F^+ \oplus F^-$$

This is the usual explicit realization of the space of complex spinors associated to $\mathbb{R}^2$ with the standard Euclidean metric. Fix, in $H \cong \mathbb{R}^2$, an orthonormal basis (f_1, f_2).

We consider the following representation ψ of the *(real) Clifford algebra* $C(H) \cong C(\mathbb{R}^2)$

$$\psi(f_1) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \psi(f_2) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Indeed, one easily checks that (the suffix "+" denoting *anticommutation*):

$$[\psi(f_1), \psi(f_1)]_+ = [\psi(f_2), \psi(f_2)]_+ = -2I, \qquad [\psi(f_1), \psi(f_2)]_+ = 0$$

The *complex* Clifford algebra

$$C(H) \otimes_{\mathbb{R}} \mathbb{C} \cong C(\mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C} \cong A(W)$$

i.e. the Canonical Anticommutation Relations (CAR) algebra $A(W)$ associated to the "one-particle space" $W := \langle e^- \rangle \cong \mathbb{C}$ generated inside F by e^-. This is an explicit realization of the so-called *Fock representation*, with vacuum vector e^+. The CAR algebra $A(W)$ is generated by the annihilation and creation operators given, respectively, by:

$$a = a(e^-) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad a^\dagger = a(e^-)^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and they are related to the Clifford generators via:

$$\psi(f_1) + i\psi(f_2) = 2ia(e^-), \quad \psi(f_1) - i\psi(f_2) = 2ia(e^-)^\dagger$$

Adapting the general formalism developed in [31] to our present case we find

$$\det\mathcal{D}|_0 = \mathbb{C}(e^+ \wedge e^-) \cong \mathbb{C}e^+ \otimes \mathbb{C}e^- \cong \mathbb{C}e^+ \otimes \mathbb{C}e^+ = \mathrm{Pf}\mathcal{D}|_0 \otimes \mathrm{Pf}\mathcal{D}|_0.$$

APPENDIX B

In this Appendix we record, again for completeness, the eigenfunctions of the operators $\{\mathcal{D}_a\}$, in the genus one case. It is clearly enough to treat the case $a = 0$. The equation

$$\mathcal{D}\psi = \begin{bmatrix} D^-\eta \\ D^+\xi \end{bmatrix} = \varepsilon \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \varepsilon\psi$$

entails

$$\mathcal{D}^2\psi = \varepsilon^2\psi$$

i.e.

$$D^-D^+\xi = \varepsilon^2\xi, \qquad D^+D^-\eta = \varepsilon^2\eta$$

whence $\varepsilon^2 = \frac{1}{4}(m_1^2 + m_2^2) = \frac{1}{4}(n_1^2 + n_2^2)$, $\varepsilon = \pm\frac{1}{2}\sqrt{m_1^2 + m_2^2}$,

$$\xi = c_1 \exp[i(m_1x + m_2y)], \quad \eta = c_2 \exp[i(n_1x + n_2y)], \qquad x, y \in [0, 2\pi]$$

and moreover, after substitution into the original equation, we have $m_j = n_j$, $j = 1, 2$, the latter stemming from $\mathbf{m} \cdot \mathbf{n} = \mathbf{m} \cdot \mathbf{m} = \mathbf{n} \cdot \mathbf{n}$. Moreover, for, say, $c_2 \neq 0$, we find $c_1/c_2 = \mp\frac{1}{\sqrt{n_1^2+n_2^2}}(n_2 + in_1)$. The harmonic spinors build up the two dimensional space

$$\mathbb{C}^2 = \mathrm{span}(e^+, e^-).$$

Acknowledgements. The author is grateful to a Referee for his/her careful reading and critical remarks. The present research was carried out within the framework of INDAM-GNSAGA activities. Financial support from Università Cattolica D1-funds is also acknowledged.

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