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WAVELET ESTIMATION OF A FUNCTION BELONGING TO LIPSCHITZ CLASS BY FIRST KIND CHEBYSHEV WAVELET METHOD

SHYAM LAL

Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi, India

VIVEK KUMAR SHARMA

Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi, India

NEHA PATEL

Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi, India

Abstract. In this paper, the function of Lipschitz class and Chebyshev Wavelet method are studied . Four new wavelet estimations of a function f belonging to Lipschitz class by Chebyshev Wavelet method are estimated. The calculated estimators are best possible in Wavelet Analysis.

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Keywords: Wavelet estimation, Chebyshev wavelet, function of Lipschitz class.

1. Introduction

There are several integral equations which are concerned with specific problems of Mathematical Physics. Sometimes these equations convert into ordinary as well as partial differential equations. The researchers like Razzaghi and Ordokhani[9,10], Alipanah and Dehghan [2], Hsiao[4], have already used orthogonal basis functions to approximate /estimate the solution of some integral equations.

E-mail addresses: shyam_lal@rediffmail.com, vivek.jaimatadi.sharma7@gmail.com, patelnehabhu@gmail.com.

Working in this direction, Akyüz- Dascioglu [1] discussed the Chebyshev polynomial solutions of systems of linear integral equations.There are few known results regarding the properties of Chebyshev Wavelet Expansions of a function $f \in L^2[0,1]$. The errors of wavelet estimation of certain functions by wavelet methods have been determined by several researchers like Bastin[3], Lal and Kumar [6, 7], and Lal and Sharma[5, 8] etc. The differentiable functions have several applications in Wavelet Analysis. Sometimes, non differentiable functions have very important role in Physics and Applied Sciences. The graph of the continuous but nowhere differentiable functions is a fractal.The Brownian path, fractional Brownian motion, typical Feynmann path and turbulent fluid are connected with irregular functions. The irregular functions are specified at every point by a local Lipschitz exponent lying between 0 and 1. This fact motivates to consider the approximation of function belonging to Lipschitz class $Lip_\alpha(0,1)$ by Chebyshev Wavelet Method. But till now no work seems to have been done for error of wavelet estimation of a function $f \in Lip_\alpha[0,1], 0 < \alpha \leq 1$, by Chebyshev Wavelet method. An attempt to make an advance study, in this direction, one of the objectives of this research paper is to estimate the error of wavelet estimation of a function f belonging to Lipschitz class by Chebyshev Wavelet Method. The estimates of this paper are new, sharper and best possible in Wavelet Analysis.

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2. Preliminaries

Chebyshev Wavelet $\psi_{n,m}(t) = \psi(k, n, m, t)$ have four arguments, $n = 1, 2, ...2^{k-1}$, k is any positive integer, m is degree of Chebyshev polynomial of first kind and t is the normalised time. The first kind Chebyshev Wavelets on the interval [0, 1) are defined by

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}} \tilde{T}_m(2^k t - \hat{n}), & \frac{\hat{n}-1}{2^k} \leq t < \frac{\hat{n}+1}{2^k}; \\ 0, & otherwise, \end{cases}$$

where

$$\tilde{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0; \\ \sqrt{\frac{2}{\pi}} T_m(t), & m \geq 1, \end{cases}$$

In the definition, the polynomials T_m are Chebyshev Polynomials of degree m over the interval $[-1, 1]$ which are defined as,

$$T_0(t) = 1, T_1(t) = t, \text{ and } T_m(t) = \cos(m \cos^{-1}(t)), \quad m = 1, 2, 3, ...$$

The set of Chebyshev Wavelets $\{\psi_{n,m}\}$ are an orthonormal set with respect to weight function $w_{k,n}(t) = w(2^k t - 2n + 1)$,where $w(t) = \frac{1}{\sqrt{1-t^2}}$.

2.1. **Chebyshev Wavelet Expansion.** A function $f \in L^2[0,1)$ is expanded as Chebyshev Wavelet series in the form of

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(t) \quad \text{(Razzaghi et.al [11])}. \tag{1}$$

where $c_{n,m} = \langle f, \psi_{n,m} \rangle_{w_{k,n}(t)} = \int_{-\infty}^{\infty} f(t)\psi_{n,m}(t)w_{k,n}(t)dt$.
If the above infinite series is truncated then (1) is written as

$$S_{2^{k-1},M}(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) = C^T\psi(t), \tag{2}$$

where $C = [c_{1,0}c_{1,1}...c_{1,M-1}...c_{2^{k-1},0}c_{2^{k-1},1}...c_{2^{k-1},M-1}]^T$ and
$\psi(t) = [\psi_{1,0}\psi_{1,1}...\psi_{1,M-1}...\psi_{2^{k-1},0}\psi_{2^{k-1},1}...\psi_{2^{k-1},M-1}]^T$.

2.2. **Wavelet Approximation.** We define $\|f\|_1 = \int_0^1 |f(x)|dx$ and $\|f\|_2 = \left\{ \int_0^1 |f(x)|^2 dx \right\}^{\frac{1}{2}}$. The Wavelet Approximation $E_{2^{k-1},M}$ of f by $S_{2^{k-1},M}$ of its Chebyshev wavelet expansion under the norm $\|.\|_2$ is defined by

$$E_{2^{k-1},M}(f) = \underset{S_{2^{k-1},M}}{inf} \|f - S_{2^{k-1},M}\|_2, \quad \text{(Zygmund [13], pp. 114)}.$$

If $E_{2^{k-1},M}(f) \to 0$ as $k \to \infty, M \to \infty$ then $E_{2^{k-1},M}(f)$ is called the best wavelet approximation of f (Zygmund [13], pp. 114) .
Similarly we can define the wavelet approximation under the norm $\|.\|_1$.

2.3. **A function of Lip_α0, 1] class.** A function $f \in Lip_\alpha[0,1]$, $0 < \alpha \leq 1$ if

$$|f(x) - f(y)| = O(|(x-y)|^\alpha)\forall x, y \in [0,1], \text{(Titchmarsh, [12], p.406)}.$$

If $0 < \alpha < \beta \leq 1$, then $Lip_\beta[0,1] \subsetneq Lip_\alpha[0,1]$.

Example 1. *Let $\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$ and $f(x) = x^{\frac{1}{3}}$, $g(x) = x^{\frac{1}{2}}$, for all $x \in [0,1]$. Then*

$$g \in Lip_\beta[0,1] \Rightarrow g \in Lip_\alpha[0,1].$$

Therefore, $f \in Lip_{\frac{1}{3}}[0,1]$ but $f \notin Lip_{\frac{1}{2}}[0,1]$. Hence, $Lip_{\frac{1}{2}}[0,1] \subsetneq Lip_{\frac{1}{3}}[0,1]$.

3. MAIN RESULTS

In this paper, Chebyshev wavelet estimations have been determined in the following forms:

Theorem 1. *If $f \in Lip_\alpha[0,1], 0 < \alpha \leq 1$ and its Chebyshev wavelet series for $m = 0$ is given by*

$$f(t) = \sum_{n=1}^{\infty} c_{n,0}\psi_{n,0}(t),$$

then Chebyshev wavelet estimation $E^{(1)}_{2^{k-1},0}(f)$ of f by its $(2^{k-1},0)^{th}$ partial sums $S_{2^{k-1},0}(t) = \sum_{n=1}^{2^{k-1}} c_{n,0}\psi_{n,0}(t)$ of Chebyshev wavelet series under the norm $\| .\|_2$ is given by

$$E^{(1)}_{2^{k-1},0}(f) = \underset{S_{2^{k-1},0}}{inf} \|f - S_{2^{k-1},0}\|_2 = O\left(\frac{1}{2^{\alpha k + \frac{1}{2}}}\right), \ 0 < \alpha \leq 1.$$

Proof. We start by

$$c_{n,0} = \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t)\psi_{n,0}(t)\frac{1}{\sqrt{1-(2^k t-\hat{n})^2}}dt, \hat{n} = 2n-1$$

$$= \frac{1}{\sqrt{\pi 2^k}} \int_0^{\pi} \left(f\left(\frac{\cos\theta + \hat{n}}{2^k}\right) - f\left(\frac{\hat{n}}{2^k}\right)\right) d\theta + \frac{1}{\sqrt{\pi 2^k}} \int_0^{\pi} f\left(\frac{\hat{n}}{2^k}\right) d\theta.$$

Therefore

$$|c_{n,0}| \leq \frac{1}{\sqrt{\pi 2^k}} \int_0^{\pi} A \ \frac{|\cos\theta|^{\alpha}}{2^{\alpha k}} d\theta + \sqrt{\frac{\pi}{2^k}} \left| f\left(\frac{\hat{n}}{2^k}\right)\right|,$$

for $f \in Lip_{\alpha}[0,1)$. Hence we have

$$|c_{n,0}|^2 \leq \frac{\pi}{2^k}\left(\frac{A^2}{2^{2\alpha k}} + \left|f\left(\frac{\hat{n}}{2^k}\right)\right|^2 + \frac{2A}{2^{\alpha k}}\left|f\left(\frac{\hat{n}}{2^k}\right)\right|\right).$$

Thus

$$-|c_{n,0}|^2 \leq -\frac{\pi}{2^k}\left|f\left(\frac{\hat{n}}{2^k}\right)\right|^2 - \frac{2\pi A}{2^{k(\alpha+1)}}\left|f\left(\frac{\hat{n}}{2^k}\right)\right|. \tag{3}$$

So we have

(4)

$$\|f\|_2^2 = \int_0^{\pi} \left|f\left(\frac{\cos\theta+\hat{n}}{2^k}\right)\right|^2 \frac{d\theta}{2^k} \leq \frac{\pi}{2^k}\frac{A^2}{2^{2\alpha k}} + \frac{\pi}{2^k}\left|f\left(\frac{\hat{n}}{2^k}\right)\right|^2 + \frac{2\pi A}{2^k}\left|f\left(\frac{\hat{n}}{2^k}\right)\right|\frac{1}{2^{\alpha k}}.$$

Since, $e_n(t) = c_{n,0}\psi_{n,0}(t) - f\chi_{[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}})}$, then

$$e_n^2(t) = c_{n,0}^2\psi_{n,0}^2 + f^2 - 2c_{n,0}\psi_{n,0}f\chi_{[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}})}.$$

Hence

$$\|e_n\|_2^2 = \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} |e_n(t)|^2 w_{k,n}(t)dt = \|f\|_2^2 - c_{n,0}^2. \tag{5}$$

By (3), (4), and (5), we have

(6)
$$\|e_n\|_2^2 = O\left(\frac{1}{2^{k(2\alpha+1)}}\right)$$
$$S_{2^{k-1},0}(t) - f(t) = \sum_{n=1}^{2^{k-1}} e_n(t)$$
$$(S_{2^{k-1},0}(t) - f(t))^2 = \left(\sum_{n=1}^{2^{k-1}} e_n(t)\right)^2 = \sum_{n=1}^{2^{k-1}} e_n^2(t) + \sum\sum_{1\le n\ne n'\le 2^{k-1}} e_n(t)e_{n'}(t)$$
$$(E^{(1)}_{2^{k-1},0}(f))^2 = \|S_{2^{k-1},0} - f\|_2^2 = \int_0^1 \left|(S_{2^{k-1},0}(t) - f(t)\right|^2 w_{k,n}(t)dt = \sum_{n=1}^{2^{k-1}} \|e_n\|_2^2.$$

By equations (6) we have

$$(E^{(1)}_{2^{k-1},0}(f)) = O\left(\frac{1}{2^{k\alpha+\frac{1}{2}}}\right).$$

This completes the proof of the theorem.

□

Theorem 2. *Let $f \in Lip_\alpha[0,1], 0 < \alpha \le 1$ and its Chebyshev wavelet series is given by*

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(t)$$

with $(2^{k-1}, M)^{th}$ partial sums

$$S_{2^{k-1},M}(t) = \sum_{n-1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t).$$

Then the Chebyshev wavelet estimation under $\|\cdot\|_2$ of f by $(2^{k-1}, M)^{th}$ partial sums of its Chebyshev wavelet series satisfies

$$E^{(2)}_{2^{k-1},M} = \underset{S_{2^{k-1},M}}{inf} \|f - S_{2^{k-1},M}\|_2 = \begin{cases} O\left(\frac{1}{M^{\frac{3}{2}}2^{\alpha k}}\right), & 0<\alpha<1, M\ge 1; \\ O\left(\frac{1}{(M-1)^{\frac{1}{2}}2^k}\right), \alpha = 1, M > 1. \end{cases}$$

Proof.

$$c_{n,m} = \langle f, \psi_{n,m} \rangle_{w_{k,n}(t)} = \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t)\psi_{n,m}(t)w_{k,n}(t)dt$$

$$= 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left\{ f(t) - f\left(\frac{2n-1}{2^k}\right)\right\} T_m(2^k t - 2n + 1)w_{k,n}(t)dt$$

$$+ 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f\left(\frac{2n-1}{2^k}\right) T_m(2^k t - 2n + 1)w_{k,n}(t)dt.$$

and

$$|c_{n,m}| \le 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}} \left| \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left\{ f(t) - f\left(\frac{2n-1}{2^k}\right)\right\} T_m(2^k t - 2n + 1)w_{k,n}(t)dt \right|$$

$$+ 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}} \left| f\left(\frac{2n-1}{2^k}\right)\right| \left| \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} T_m(2^k t - 2n + 1)w_{k,n}(t)dt \right| = I_1 + I_2. \tag{7}$$

Thus,

$$I_1 \le \frac{2^{\frac{k}{2}}}{2^{\alpha k}}\sqrt{\frac{2}{\pi}} \int_0^{\pi} |\cos\theta|^{\alpha} \left| T_m(\cos\theta)\frac{1}{\sqrt{1-\cos^2\theta}} \frac{-\sin\theta d\theta}{2^k} \right|,$$

$$= \frac{1}{2^{\frac{2\alpha k + k}{2}}}\sqrt{\frac{2}{\pi}} \int_0^{\pi} |\cos\theta|^{\alpha} |\cos m\theta|\, d\theta. \tag{8}$$

For $\alpha = 1$,

$$I_1 \le \frac{1}{2^{\frac{3k}{2}}}\sqrt{\frac{2}{\pi}} \int_0^{\pi} |\cos\theta \cos m\theta|\, d\theta. \tag{9}$$

Since,

$$\int_0^{\pi} \cos\theta \cos m\theta d\theta \le \frac{m|\cos m\pi||\sin m\pi| + |\sin\pi||\cos m\pi|}{m^2-1}(\pi - 0)$$
$$= \frac{(m+1)\pi}{m^2-1} = \frac{\pi}{m-1}. \tag{10}$$

By equations (9) and (10),

$$I_1 \le \frac{\sqrt{2\pi}}{2^{\frac{3k}{2}}(m-1)}. \tag{11}$$

For $0 < \alpha < 1$, since $|\cos\theta| \leq e^{\theta}$ $\forall\theta \in [0,\pi]$, using this in the equation (8) we have

$$I_1 \leq \frac{1}{2^{\frac{2\alpha k+k}{2}}}\sqrt{\frac{2}{\pi}}\int_0^{\pi}\left|e^{\alpha\theta}\cos m\theta\right| d\theta.$$

$$J = \int_0^{\pi} e^{\alpha\theta}\cos m\theta d\theta = \frac{\alpha(e^{\alpha}\pi\cos(m\pi)-1)}{m^2+\alpha^2}. \tag{12}$$

$$\int_0^{\pi}|e^{\alpha\theta}\cos m\theta|d\theta \leq \frac{\alpha\pi(e^{\alpha\pi}+1)}{m^2+\alpha^2}.$$

By equations above we have we have

$$\begin{aligned} I_1 &\leq \frac{1}{2^{\frac{2\alpha k+k}{2}}}\sqrt{\frac{2}{\pi}}\frac{\alpha\pi(e^{\alpha\pi}+1)}{m^2+\alpha^2}, \\ I_2 &= 2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}\left|f\left(\frac{2n-1}{2^k}\right)\right|\left|\int_0^{\pi} T_m(\cos\theta)\frac{1}{2^k}d\theta\right| = 0 \end{aligned} \tag{13}$$

Collecting equations above we get

$$\begin{aligned} |c_{n,m}| &\leq \frac{\sqrt{2\pi}}{2^{\frac{3k}{2}}(m-1)}, \quad \text{for} \quad \alpha = 1 \\ |c_{n,m}| &\leq \frac{1}{2^{\frac{(2\alpha+1)k}{2}}}\sqrt{\frac{2}{\pi}}\frac{\alpha\pi(e^{\alpha\pi}+1)}{m^2+\alpha^2}, \quad \text{for} \quad 0 < \alpha < 1. \end{aligned} \tag{14}$$

Therefore,

$$\begin{aligned} (f(t) - S_{2^{k-1},M}(t))^2 = &\sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty} c_{n,m}^2\psi_{n,m}^2(t) \\ &+ \sum_{n=1}^{2^{k-1}}\sum\sum_{M\leq m\neq m'<\infty} c_{n,m}c_{n,m'}\psi_{n,m}(t)\psi_{n,m'}(t) \\ &+ \sum\sum_{1\leq n\neq n'\leq 2^{k-1}}\sum_{m=m'=M}^{\infty} c_{n,m}c_{n',m'}\psi_{n,m}(t)\psi_{n',m'}(t) \\ &+ \sum\sum_{1\leq n\neq n'\leq 2^{k-1}}\sum\sum_{M\leq m\neq m'<\infty} c_{n,m}c_{n',m'}\psi_{n,m}(t)\psi_{n',m'}(t). \end{aligned}$$

Hence,

$$\|f - S_{2^{k-1},M}\|_2^2 = \int_0^1 \left|f(t) - S_{2^{k-1},M}(t)\right|^2 w_{k,n}(t)dt = \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}|c_{n,m}|^2.$$

By equations above

$$\|f - S_{2^{k-1},M}\|_2^2 \leq \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\frac{2\pi}{2^{3k}(m-1)^2} \leq \frac{2\pi}{2^{2k}}\left(\frac{1}{M-1}\right), M > 1, \alpha = 1.$$

Therefore,

$$E^{(2)}_{2^{k-1},M}(f) = \underset{S_{2^{k-1},M}}{inf} \|f - S_{2^k-1,M}\|_2 = O\left(\frac{1}{2^k}\frac{1}{(M-1)^{\frac{1}{2}}}\right), \tag{15}$$

for $M > 1$ and $\alpha = 1$. Also by equations above

$$E^{(2)}_{2^{k-1},M}(f) = \underset{S_{2^{k-1},M}}{inf} \|f - S_{2^{k-1},M}\|_2 = O\left(\frac{1}{2^{\alpha k} M^{\frac{3}{2}}}\right),$$

for $M \geq 1$ and $0 < \alpha < 1$. This completes the proof. □

Theorem 3. *If $f \in Lip_\alpha[0,1], 0 < \alpha \leq 1$ then Chebyshev wavelet estimation $E^{(3)}_{2^{k-1},0}(f)$ of a function f by $S_{2^{k-1},0}(t)$, for $m = 0$ under the norm $\|.\|_1$ is given by*

$$E^{(3)}_{2^{k-1},0}(f) = \underset{S_{2^{k-1},0}}{inf} \|f - S_{2^{k-1},0}\|_1 = O\left(\frac{1}{2^{\alpha(k-1)}}\right), 0 < \alpha \leq 1.$$

Proof. By Mean Value Theorem of integral calculus we have

$$c_{n,0} = \sqrt{\frac{\pi}{2^k}} f\left(\frac{\cos\theta_n + \hat{n}}{2^k}\right),$$

for $\theta_n \in [0, \pi)$. Since $e_n(t) = c_{n,0}\psi_{n,0}(t) - f(t)$, then

$$\|e_n\|_1 = \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} |c_{n,0}\psi_{n,0}(t) - f(t)|\, w_{k,n}(t)dt = O\left(\frac{1}{2^k 2^{\alpha(k-1)}}\right).$$

Hence,

$$E^{(3)}_{2^{k-1},0} = \sum_{n=1}^{2^{k-1}} \|e_n\|_1 = O\left(\frac{1}{2^{\alpha(k-1)}}\right).$$

This completes the proof. □

Theorem 4. *If a function $f \in Lip_\alpha[0,1]$, for $0 < \alpha \leq 1$ then Chebyshev Wavelet estimation of f by $(2^{k-1}, M)^{th}$ partial sums , $S_{2^{k-1},M}$ of its Chebyshev wavelet series under the norm $\|\cdot\|_1$ is given by*

$$E^{(4)}_{2^{k-1},M} = \underset{S_{2^{k-1},M}}{inf} \|f - S_{2^{k-1},M}\|_1 = O\left(\frac{1}{M 2^{\alpha k}}\right), M \geq 1, 0 < \alpha \leq 1.$$

Proof. Following the proof of the Theorem 1

$$\begin{aligned}\|f - S_{2^{k-1},M}\|_1 &\leq \sum_{n=1}^{2^{k-1}} \sum_{m=M}^{\infty} |c_{n,m}| \|\psi_{n,m}\|_1 \\ &\leq 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} \sum_{n=1}^{2^{k-1}} \sum_{m=M}^{\infty} |c_{n,m}| \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} 1dt, = 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} \sum_{m=M}^{\infty} |c_{n,m}|,\end{aligned}$$

because $|T_m(2^k t - 2n + 1)| \leq 1$, for all $t \in [0, 1)$. Now we have

$$\|f - S_{2^{k-1},M}\|_1 \leq \frac{2\alpha(e^{\alpha\pi} + 1)}{2^{\alpha k}} \frac{2}{M} = \frac{4\alpha(e^{\alpha\pi} + 1)}{2^{\alpha k}} \frac{1}{M}, 0 < \alpha < 1.$$

Thus,

$$E^{(4)}_{2^{k-1},M}(f) = \underset{S_{2^{k-1},M}}{inf} \|f - S_{2^{k-1},M}\|_1 = O\left(\frac{1}{2^{\alpha k}}\frac{1}{M}\right), \quad M \geq 1, 0 < \alpha \leq 1.$$

This completes the proof.

□

4. Numerical Examples of Wavelet Approximation

In this section Chebyshev wavelet approximation of function

$$f(t) = \begin{cases} \sqrt{1-t}, & \forall t \in [0,1] \\ 0, & otherwise. \end{cases}$$

for $m = 0$ has been explained by graphs of concerned function and corresponding approximations $E_2^{k-1}(f)$.

$S_{2^{k-1},0}$ for $m = 0$ and $k = 1, 2, 3, 4$ are calculated and are given as

$$S_{2^0,0}(t) = \begin{cases} \frac{2}{\pi}, & 0 \leq t < 1, \\ 0, & otherwise. \end{cases} \quad S_{2^1,0}(t) = \begin{cases} \frac{1.52403968}{\sqrt{\pi}}, & 0 \leq t < \frac{1}{2}, \\ \frac{0.79788456}{\sqrt{\pi}}, & \frac{1}{2} \leq t < 1, \\ 0, & otherwise. \end{cases}$$

$$S_{2^2,0}(t) = \begin{cases} \frac{1.655853816}{\sqrt{\pi}}, & 0 \leq t < \frac{1}{4}, \\ \frac{1.39771223}{\sqrt{\pi}}, & \frac{1}{4} \leq t < \frac{1}{2}, \\ \frac{1.077658793}{\sqrt{\pi}}, & \frac{1}{2} \leq t < \frac{3}{4}, \\ \frac{0.564190157}{\sqrt{\pi}}, & \frac{3}{4} \leq t < 1, \\ 0, & otherwise. \end{cases} \quad S_{2^3,0}(t) = \begin{cases} \frac{1.71570052}{\sqrt{\pi}}, & 0 \leq t < \frac{1}{8}, \\ \frac{1.59707966}{\sqrt{\pi}}, & \frac{1}{8} \leq t < \frac{2}{8}, \\ \frac{1.468881684}{\sqrt{\pi}}, & \frac{2}{8} \leq t < \frac{3}{8}, \\ \frac{1.328311028}{\sqrt{\pi}}, & \frac{3}{8} \leq t < \frac{4}{8}, \\ \frac{1.17086546}{\sqrt{\pi}}, & \frac{4}{8} \leq t < \frac{5}{8}, \\ \frac{.988331796}{\sqrt{\pi}}, & \frac{5}{8} \leq t < \frac{6}{8}, \\ \frac{0.76201984}{\sqrt{\pi}}, & \frac{6}{8} \leq t < \frac{7}{8}, \\ \frac{0.398938452}{\sqrt{\pi}}, & \frac{7}{8} \leq t < 1, \\ 0, & otherwise. \end{cases}$$

The errors $\|e_n\|_2^2$ and $(E_{2^{k-1}}f)^2$ are calculated for $k = 1, 2, 3, 4, 5$ and also the graph of $S_{2^{k-1},0}$ and $f(t)$ has been plotted for $k = 1, 2, 3, 4$.

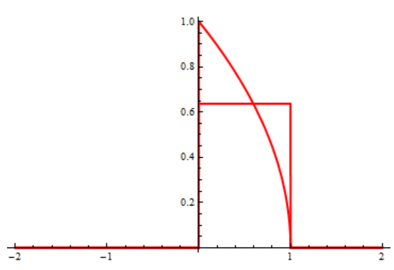

FIGURE 1. *
Graph of $S_{2^0,0}$ and the function $f(t)$

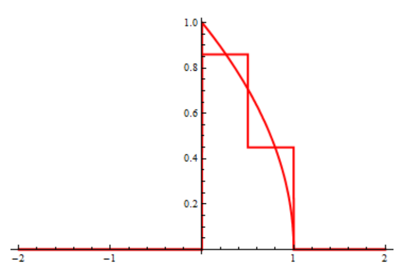

FIGURE 2. *
Graph of $S_{2^1,0}$ and the function $f(t)$

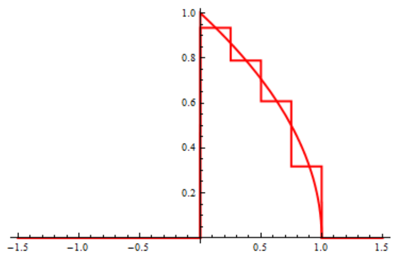

FIGURE 3. *
Graph of $S_{2^2,0}$ and the function $f(t)$

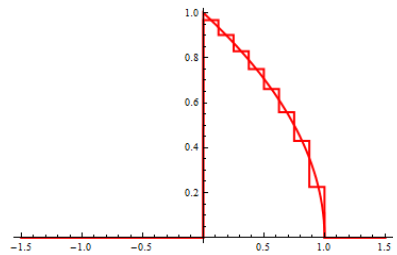

FIGURE 4. *
Graph of $S_{2^3,0}$ and the function $f(t)$

		$\|e_n\|_2^2$	$(E_2^{k-1}f)^2$
k= 1	n=1	0.14877893	0.14877893
k= 2	n=1	0.008374386	
	n=2	0.037194597	0.04556893
k= 3	n=1	0.000880215	
	n=2	0.001237026	0.013509406
	n=3	0.002093596	
	n=4	0.009298569	
k=4	n=1	0.000100927	
	n=2	0.000117536	
	n=3	0.000139472	
	n=4	0.000170979	0.003906621
	n=5	0.000220054	
	n=6	0.000309401	
	n=7	0.000523399	
	n=8	0.002324853	
k =5	n=1	0.000012479	
	n=2	0.000013612	
	n=3	0.00001415	
	n=4	0.000015266	
	n=5	0.000016383	
	n=6	0.000018671	
	n=7	0.000020302	
	n=8	0.000022693	0.00110127
	n=9	0.000025232	
	n=10	0.000029384	
	n=11	0.000034868	
	n=12	0.000042745	
	n=13	0.000055014	
	n=14	0.000077312	
	n=15	0.00013085	
	n=16	.000581166	

TABLE 1. *

Wavelet approximation errors for different values of k

Remark 1. *Let $f \in L^2[0,1]$*

$$\int_0^1 |f(x)|dx \leq \left(\int_0^1 |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 dx\right)^{\frac{1}{2}}, \text{ by Hölder's inequality}$$
$$= ||f||_2 < \infty$$

therefore, $f \in L^1[0,1]$. Hence $L^2[0,1] \subseteq L^1[0,1]$. Let

$$f_1(t) = \begin{cases} \frac{1}{\sqrt{t}}, & \forall t \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

Then $\int_0^1 f_1 dx = \int_0^1 \frac{1}{\sqrt{x}} dx = 2$ and $\int_0^1 f_1^2 dx = \int_0^1 \frac{1}{x} dx = \infty$. Thus $f_1 \in L^1[0,1)$ and $f_1 \notin L^2[0,1)$. Therefore, $L^2[0,1) \subsetneq L^1[0,1)$.

Remark 2. *Lipschitz regularity has been used in the evaluation of following part of the estimators:*

(i) $c_{n,0}$, $||f||_2$ for the estimator $E^{(1)}_{2^{k-1},0}(f)$ of Theorem 1.

(ii) $c_{n,m}$ for the estimator $E^{(2)}_{2^{k-1},M}(f)$ of Theorem 2.

(iii) $c_{n,0}$ for the estimator $E^{(3)}_{2^{k-1},0}(f)$ of Theorem 3.

(iv) $c_{n,m}$ for the estimator $E^{(4)}_{2^{k-1},M}(f)$ of Theorem 4.

Lipschitz regularity is already mentioned in the manuscript where it is applied. Thus, Lipschitz regularity has major role in obtaining the estimators $E^{(1)}_{2^{k-1},0}(f)$, $E^{(2)}_{2^{k-1},M}(f)$, $E^{(3)}_{2^{k-1},0}(f)$, $E^{(4)}_{2^{k-1},M}(f)$.

Remark 3. *The following are worth noting:*

(i) $E^{(1)}_{2^{k-1},0}(f) = O\left(\frac{1}{2^{\alpha k+\frac{1}{2}}}\right)$, and $E^{(3)}_{2^{k-1},0}(f) = O\left(\frac{1}{2^{\alpha(k-1)}}\right)$, for $0 < \alpha \leq 1$. Since

$$\frac{1}{2^{\alpha k+\frac{1}{2}}} \leq \frac{1}{2^{\alpha(k-1)}},$$

therefore, $E^{(1)}_{2^{k-1},0}(f)$ is sharper than $E^{(3)}_{2^{k-1},0}(f)$.

(ii) $E^{(2)}_{2^{k-1},M} = O\left(\frac{1}{M^{\frac{3}{2}}2^{\alpha k}}\right)$, for $0 < \alpha < 1$, and $M \geq 1$. Also, $E^{(4)}_{2^{k-1},M} = O\left(\frac{1}{M2^{\alpha k}}\right)$, for $M \geq 1$ and $0 < \alpha \leq 1$. Since

$$\frac{1}{M^{\frac{3}{2}}2^{\alpha k}} \leq \frac{1}{M2^{\alpha k}},$$

therefore, $E^{(2)}_{2^{k-1},M}(f)$ is sharper than $E^{(4)}_{2^{k-1},M}(f)$. Hence, the estimator $E^{(1)}_{2^{k-1},0}(f)$ and $E^{(2)}_{2^{k-1},M}(f)$ of $L^2[0,1)$ are sharper than estimator $E^{(3)}_{2^{k-1},0}(f)$ and $E^{(4)}_{2^{k-1},M}(f)$ respectively of $L^1[0,1)$.

5. Final remarks

By the estimates of the Theorems 1, 2, 3 and 4,

$$E^{(1)}_{2^{k-1},0}, E^{(2)}_{2^{k-1},M}, E^{(3)}_{2^{k-1},0} \text{and } E^{(4)}_{2^{k-1},M} \to 0 \text{ as } M \to \infty, k \to \infty$$

Therefore the estimates $E^{(1)}_{2^{k-1},0}, E^{(2)}_{2^{k-1},M}, E^{(3)}_{2^{k-1},0}$ and $E^{(4)}_{2^{k-1},M}$ calculated in this research paper are best possible.

Since

$$M^{\frac{3}{2}}2^{\alpha k} \geq M2^{\alpha k}$$

implies

$$\frac{1}{M^{\frac{3}{2}}2^{\alpha k}} \leq \frac{1}{M2^{\alpha k}},$$

for $0 < \alpha < 1$, and $M \geq 1$ therefore the estimator $E^{(2)}_{2^{k-1},M}$ is sharper and better than the estimator $E^{(4)}_{2^{k-1},M}$. Thus, the estimators are obtained generally in $\|.\|_2$ norm. The main importance of these estimators is that they depend on $0 < \alpha \leq 1$ if the function $f \in Lip_\alpha[0,1]$. The estimate $E^{(2)}_{2^{k-1},M}$ for $\alpha = 1$, can not be obtained from the estimate $E^{(2)}_{2^{k-1},M}$ for $0 < \alpha < 1$ taking $\alpha = 1$ in this case. Thus the estimate for $\alpha = 1$ is independent of the estimate for $0 < \alpha < 1$.

REFERENCES

[1] Akyüz-DaşcıoÇğlu, Ayşegül, *Chebyshev polynomial solutions of systems of linear integral equations*, Applied Mathematics and Computation **151** (2004), no. 1, 221–232.

[2] Alipanah, Amjad and Dehghan, Mehdi, *Numerical solution of the nonlinear fredholm integral equations by positive definite functions*, Applied Mathematics and Computation **190** (2007), no. 2, 1754–1761.

[3] Bastin, Françoise, *A riesz basis of wavelets and its dual with quintic deficient splines*, Note di Matematica **25** (2006), no. 1, 55–62.

[4] Hsiao, Chun-Hui, *Hybrid function method for solving fredholm and volterra integral equations of the second kind*, Journal of Computational and Applied Mathematics **230** (2009), no. 1, 59–68.

[5] Lal, S. and Sharma, V. K., *On wavelet approximation of a function by Legendre wavelet methods*, Functional Analysis, Approximation and Computation **9** (2017), no. 2, 11–19.

[6] Lal, Shyam and Kumar, Manoj, *Approximation of functions of space $l^2(\mathbb{R})$ by wavelet expansions*, Lobachevskii Journal of Mathematics **34** (2013), no. 2, 163–172.

[7] Lal, Shyam and Kumar, Susheel, *Best wavelet approximation of functions belonging to generalized lipschitz class using haar scaling function*, Thai Journal of Mathematics **15** (2017), no. 2, 409–419.

[8] Lal, Shyam and Sharma, Vivek Kumar, *On the approximation of a continuous function f (x, y) by its two dimensional legendre wavelet expansion*, International Journal of Computer Applications **143** (2016), no. 6.

[9] Razzaghi, M and Ordokhani, Y, *Solution of nonlinear volterra-hammerstein integral equations via rationalized haar functions*, Mathematical Problems in Engineering **7** (2001), no. 2, 205–219.

[10] Razzaghi, Mohsen and Ordokhani, Yadollah, *A rationalized haar functions method for nonlinear fredholm-hammerstein integral equations*, International journal of computer mathematics **79** (2002), no. 3, 333–343.

[11] Razzaghi, Mohsen and Yousefi, S, *The legendre wavelets operational matrix of integration*, International Journal of Systems Science **32** (2001), no. 4, 495–502.

[12] Titchmarsh, E.C.., *The theory of functions* (1939).

[13] Zygmund, A.., *Trigonometric series, reprinted with corrections and some additions, volume i, cambridge univ*, Pres, 1959.