# ON THE DISTRIBUTION OF THE VALUES OF ADDITIVE FUNCTIONS OVER INTEGERS WITH A FIXED NUMBER OF DISTINCT PRIME DIVISORS

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ABSTRACT. We study the distribution of the values of certain additive functions restricted to those integers with a fixed number of prime divisors.

## 1. INTRODUCTION

We study the distribution of the values of certain additive functions restricted to those integers with a fixed number of prime divisors.

Given an additive function $f$ for which there exists a real number $C > 0$ such that $|f(p^a)| < C$ for all prime powers $p^a$, we let

$$A_x = \sum_{p \le x} \frac{f(p)}{p}$$

and we let $f^*$ be the additive function (which depends on $x$) defined on prime powers by $f^*(p^a) = f(p^a) - \dfrac{a}{x_2} A_x$, where $x_2 = \log\log x$. Let

$$B_x = \sqrt{\sum_{p \leq x} \frac{(f^*(p))^2}{p}}$$

and assume that $B_x \to \infty$. For each integer $k \geq 1$, let

$$\xi_{k,x} := \frac{k}{x_2}, \qquad \wp_k := \{n \in \mathbb{N} : \omega(n) = k\}, \qquad \pi_k(x) := \#\{n \leq x : n \in \wp_k\}.$$

Finally, let $\delta < \frac{1}{2}$ be a fixed positive number. Then, we prove that

$$\lim_{x \to \infty} \max_{\substack{k \\ \xi_{k,x} \in [\delta, 2-\delta]}} \max_{y \in \mathbb{R}} \left| \frac{1}{\pi_k(x)} \# \left\{ n \leq x : n \in \wp_k, \ \frac{f^*(n)}{B_x\sqrt{\xi_{k,x}}} < y \right\} - \Phi(y) \right| = 0,$$

where

(1.1) $$\Phi(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-u^2/2} \, du.$$

We also establish a result concerning the distribution of $\omega(\varphi_\ell(n))$ where $\omega(m)$ stands for the number of distinct prime factors of $m \geq 2$ (with $\omega(1) = 0$) and $\varphi_\ell$ stands for the $\ell$-th iterate of the Euler $\varphi$-function.

## 2. Notations

For each integer $n \geq 2$, $\Omega(n)$ stand for the number of prime divisors of $n$ counting their multiplicity, setting $\Omega(1) = 0$. Let also $p(n)$ and $P(n)$ stand for the smallest and largest prime factors of $n \geq 2$, with $p(1) = P(1) = 1$.

We shall use the notations $x_1 = \log x$, $x_2 = \log\log x$, and so on.

For every positive integers $k$ and $D$, let us further set

$$\begin{aligned}
\wp_k &:= \{n \in \mathbb{N} : \omega(n) = k\}, \\
\pi_k(x) &:= \#\{n \leq x : n \in \wp_k\}, \\
\mathcal{N}_k &:= \{n \in \mathbb{N} : \Omega(n) = k\}, \\
N_k(x) &:= \#\{n \leq x : n \in \mathcal{N}_k\}, \\
\pi_k(x|D) &:= \#\{n \leq x : (n, D) = 1, \ n \in \wp_k\}, \\
N_k(x|D) &:= \#\{n \leq x : (n, D) = 1, \ n \in \mathcal{N}_k\}, \\
\xi_{k,x} &:= \frac{k}{x_2}.
\end{aligned}$$

Let $\Phi$ be the standard Gaussian law defined above in (1.1). We also write $\psi(t)$ for the characteristic function of the Gaussian law, that is,

$$\psi(t) := e^{-t^2/2} \qquad (t \in \mathbb{R}).$$

We shall also be using the two sequences of integers

(2.1) $$a_\ell = \frac{1}{(\ell+1)!} \quad \text{and} \quad b_\ell = \frac{1}{\sqrt{2\ell+1}} \cdot \frac{1}{\ell!} \qquad (\ell = 1, 2, \ldots).$$

Throughout this paper, the letters $c$ and $C$ always denote positive constants, but not necessarily the same at each occurrence.

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### 3. Main results

**Theorem 1.** *Let $f$ be an additive function for which there exists a real number $C > 0$ such that $|f(p^a)| < C$ for all prime powers $p^a$. Let $A_x = \sum_{p \le x} \dfrac{f(p)}{p}$. Let $f^* = f_x^*$ be the additive function defined on prime powers by*

$$f^*(p^a) = \begin{cases} f(p^a) - \frac{a}{x_2} A_x & \text{if } p^a \le x, \\ 0 & \text{otherwise} . \end{cases}$$

*Further set*

$$B_x = \sqrt{\sum_{p \le x} \frac{(f^*(p))^2}{p}}$$

*and assume that $B_x \to \infty$ as $x \to \infty$. Then, given an arbitrary positive real number $\delta < \frac{1}{2}$,*

$$\lim_{x \to \infty} \max_{\substack{k \\ \xi_{k,x} \in [\delta, 2-\delta]}} \max_{y \in \mathbb{R}} \left| \frac{1}{\pi_k(x)} \# \left\{ n \le x : n \in \wp_k, \ \frac{f^*(n)}{B_x \sqrt{\xi_{k,x}}} < y \right\} - \Phi(y) \right| = 0.$$

Let us add that in 2008, Kátai and Subbarao [3] proved the following result.

**Theorem A.** *With the notations of Theorem 1, we have*

$$\lim_{x \to \infty} \max_{\xi_{k,x} \in [\delta, 2-\delta]} \max_{y \in \mathbb{R}} \left| \frac{1}{N_k(x)} \# \left\{ n \le x : n \in \mathcal{N}_k, \ \frac{f^*(n)}{B_x \sqrt{\xi_{k,x}}} < y \right\} - \Phi(y) \right| = 0.$$

**Theorem 2.** *Let $a_\ell$ and $b_\ell$ be the two sequences defined in (2.1). Let $\xi = \xi_{k,x}$ and assume that $\ell$ is fixed. Setting*

$$s_\xi(n) := \frac{\omega(\varphi_\ell(n)) - a_\ell \xi x_2^{\ell+1}}{b_\ell \sqrt{\xi} x_2^{\ell + \frac{1}{2}}},$$

*then, given an arbitrary positive real number $\delta < \frac{1}{2}$,*

(3.1) $$\lim_{x \to \infty} \max_{z \in \mathbb{R}} \max_{\xi \in [\delta, 2-\delta]} \left| \frac{1}{N_k(x)} \# \{ n \le x : n \in \mathcal{N}_k, \ s_\xi(n) < z \} - \Phi(z) \right| = 0$$

*and*

(3.2) $$\lim_{x \to \infty} \max_{z \in \mathbb{R}} \max_{\xi \in [\delta, 2-\delta]} \left| \frac{1}{\pi_k(x)} \# \{ n \le x : n \in \wp_k, \ s_\xi(n) < z \} - \Phi(z) \right| = 0.$$

### 4. An appropriate estimate for $\pi_k(x|D)$

As a preliminary result to be used in the proof of Theorem 1, we will show the following lemma.

**Lemma 1.** *Given a positive number $\delta < \frac{1}{2}$, then, as $x \to \infty$, we have, uniformly for $\xi \in [\delta, 2 - \delta]$,*

(4.1) $$\pi_k(x|D) = (1 + o(1))\pi_k(x) \cdot \prod_{p|D} \left( 1 - \frac{\xi}{p \left( 1 - \frac{1-\xi}{p} \right)} \right).$$

*Proof.* It is well known that

$$\pi_k(x) = F(\xi) \frac{x}{x_1} \frac{x_2^{k-1}}{(k-1)!} \left( 1 + O\left( \frac{1}{x_2} \right) \right),$$

where

$$F(z) = \frac{1}{\Gamma(z+1)} \prod_p \left( 1 + \frac{z}{p-1} \right) \left( 1 - \frac{1}{p} \right)^z.$$

(see for instance the classical paper of Selberg [4]).

Given an integer $D \geq 2$, let $\mathcal{B}_D$ stand for the multiplicative semigroup generated by the prime divisors of $D$.

Let us first write

(4.2) $$\sum_{\substack{n=1 \\ (n,D)=1}}^{\infty} \frac{z^{\omega(n)}}{n^s} = \left\{ \sum_{n=1}^{\infty} \frac{z^{\omega(n)}}{n^s} \right\} \Lambda_k(s,z),$$

where

$$\Lambda_k(s,z) = \prod_{p|D} \left( 1 + \frac{z}{p^s} + \frac{z}{p^{2s}} + \ldots \right)^{-1} = \prod_{p|D} \left( 1 - \sum_{\ell=1}^{\infty} \frac{z(1-z)^{\ell-1}}{p^{\ell s}} \right) = \sum_{\substack{m=1 \\ m \in \mathcal{B}_D}}^{\infty} \frac{E(m)}{m^s},$$

where $E(m)$ is a multiplicative function defined on the set $\mathcal{B}_D$ by $E(1) = 1$, $E(p) = -z$, $E(p^\ell) = -z(1-z)^{\ell-1}$ for each $\ell \geq 2$ and each $p \in \mathcal{B}_D$.

Now, let us write each positive integer $m$ as $m = MR$, where $M$ is the squarefull part of $m$ and where $R$ is squarefree. Then, clearly,

$$E(M) = (-z)^{\omega(M)}(1-z)^{\Omega(M)-\omega(M)}, \quad E(R) = (-z)^{\omega(R)},$$

implying that if we set $\Delta(m) = \Omega(m) - \omega(m)$, then

$$E(m) = (-z)^{\omega(m)}(1-z)^{\Delta(m)} = \sum_{\nu=0}^{\Delta(m)} \binom{\Delta(m)}{\nu} (-1)^{\nu+\omega(m)} z^{\nu+\omega(m)},$$

so that it follows from (4.2) that

(4.3) $$\pi_k(x|D) = \sum_{m \in \mathcal{B}_D} \sum_{\nu=0}^{\Delta(m)} (-1)^{\nu+\omega(m)} \binom{\Delta(m)}{\nu} \pi_{k-(\omega(m)+\nu)} \left( \frac{x}{m} \right).$$

Given a fixed positive real number $\delta < 1/2$, then it is easy to prove that, uniformly for $k \in [\delta x_2, (2-\delta)x_2]$, we have

(4.4) $$\frac{\pi_{k-(\omega(m)+\nu)} \left( \frac{x}{m} \right)}{\pi_k(x)} = (1 + \varepsilon_2(x)) \xi^{\omega(m)+\nu} \qquad (x \to \infty),$$

where $\varepsilon_2(x) \to 0$ as $x \to \infty$.

Using (4.4) in (4.3), we get

$$\begin{aligned}
\frac{\pi_k(x|D)}{\pi_k(x)} &= (1 + \varepsilon_2(x)) \sum_{m \in \mathcal{B}_D} \frac{1}{m} \sum_{\nu=0}^{\Delta(m)} (-1)^{\nu+\omega(m)} \binom{\Delta(m)}{\nu} \xi^{\omega(m)+\nu} \\
&= (1 + \varepsilon_2(x)) \sum_{m \in \mathcal{B}_D} \frac{(-\xi)^{\omega(m)}}{m} (1-\xi)^{\Delta(m)}
\end{aligned}$$

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$$= (1 + \varepsilon_2(x)) \prod_{p|D} \left( 1 + \frac{-\xi}{p} + \frac{-\xi(1-\xi)}{p^2} + \cdots \right)$$

$$= (1 + \varepsilon_2(x)) \prod_{p|D} \left( 1 - \frac{\xi}{p \left( 1 - \frac{1-\xi}{p} \right)} \right),$$

which proves (4.1), thus completing the proof of Lemma 1. □

## 5. Proof of Theorem 1

From the identity

$$\sum_{\substack{n=1 \\ (n,D)=1}}^{\infty} \frac{z^{\Omega(n)}}{n^s} = \sum_{n=1}^{\infty} \frac{z^{\Omega(n)}}{n^s} \times \prod_{p|D} \left( 1 - \frac{z}{p^s} \right) \qquad (z \in \mathbb{C}, |z| < 2, s > 1),$$

it follows that

(5.1) $$N_k(x|D) = \sum_{d|D} \mu(d) N_{k-\Omega(d)} \left( \frac{x}{d} \right).$$

Let $w_x$ be a function which tends to $+\infty$ (as $x \to \infty$), but slowly enough so that $w_x / \log B_x \to \infty$ as $x \to \infty$.

Let $\nu(n) = \dfrac{f^*(n)}{B_x \sqrt{\xi}}$ and let us introduce the additive function $\nu^*$ defined on prime powers by

$$\nu^*(p^a) = \begin{cases} 0 & \text{if } p \le w_x, \\ \nu(p^a) & \text{otherwise.} \end{cases}$$

We further introduce the functions $g(n) = e^{i\tau\nu(n)}$ and $g^*(n) = e^{i\tau\nu^*(n)}$. It follows from these definitions that

(5.2) $$\max_{n \le x} |g(n) - g^*(n)| \le c|\tau| \max_{P(m) \le w_x} |\nu(m)| \to 0 \qquad \text{as } x \to \infty$$

uniformly for $\xi \in [\delta, 2 - \delta]$.

Assume that $P(D) \le w_x$. Then $g^*(d) = 1$ for every $d \in \mathcal{B}_D$ and therefore

$$\sum_{\substack{n=1 \\ (n,D)=1}}^{\infty} \frac{g^*(n) z^{\Omega(n)}}{n^s} = \sum_{n=1}^{\infty} \frac{g(n) z^{\Omega(n)}}{n^s} \times \prod_{p|D} \left( 1 - \frac{z}{p^s} \right) \qquad (z \in \mathbb{C}, \ |z| < 2, \ s > 1).$$

Let

$$M_k(x|D) := \sum_{\substack{n \le x \\ (n,D)=1 \\ \Omega(n)=k}} g^*(n); \qquad M_k(x|1) = M_k(x).$$

Then

(5.3) $$M_k(x|D) = \sum_{d|D} \mu(d) M_{k-\Omega(d)} \left( \frac{x}{d} \right).$$

Using Theorem 3 of Kátai and Subbarao [3], we obtain that

$$\lim_{x \to \infty} \sup_{\xi \in [\delta, 2-\delta]} \left| \frac{1}{N_k(x)} \sum_{\substack{n \le x \\ n \in \mathcal{N}_k}} g(n) - \psi(\tau) \right| = 0$$

uniformly for $\tau \in [-C, C]$, where $C$ is a positive constant depending only on $g$ (that is, on $f^*$), implying that, in light of (5.2),

$$(5.4) \qquad \lim_{x \to \infty} \sup_{\xi \in [\delta, 2-\delta]} \left| \frac{1}{N_k(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k}} g^*(n) - \psi(\tau) \right| = 0.$$

Now, for each divisor $d$ in (5.3), we have

$$\frac{k - \Omega(d)}{x_2} \geq \delta - \frac{\Omega(d)}{x_2} \geq \frac{\delta}{2} \qquad \text{provided } x \text{ is large enough.}$$

Thus, applying (5.4) with $\delta/2$ in place of $\delta$, we get that

$$(5.5) \qquad M_{k-\Omega(d)}\left(\frac{x}{d}\right) = (1 + o(1))N_k(x|D)\psi(\tau) \qquad (x \to \infty)$$

uniformly as $D$ runs over the integers satisfying $P(D) \leq w_x$ and $|\tau| \leq C$.

Now, for $Y \geq 2$, let $Q_Y$ stand for $\prod_{p \leq Y} p$ and $\mathcal{B}_Y$ for the multiplicative semigroup generated by $\{p \in \wp : p \leq Y\}$.

Observe that

$$(5.6) \qquad \pi_k(x) = \sum_{d \in \mathcal{B}_Y} \pi_{k-\omega(d)}\left(\frac{x}{d}\Big|Q_Y\right).$$

Now split the right hand side of (5.6) as follows:

$$(5.7) \qquad \pi_k(x) = \sum_{d \leq Y^Y} + \sum_{d > Y^Y} = \Sigma_1 + \Sigma_2,$$

say. First, we have, using the Hardy-Ramanujan inequality $\pi_k(x) \leq C \dfrac{x}{x_1} \dfrac{(x_2 + c)^{k-1}}{(k-1)!}$ uniform in $k$ (see Hardy and Ramanujan [2]),

$$\Sigma_2 \ \leq \ \sum_{\substack{Y^Y \leq d \leq \sqrt{x} \\ d \in \mathcal{B}_Y}} \frac{x}{dx_1} \frac{(x_2 + c)^{k-\omega(d)-1}}{(k-\omega(d)-1)!} + O\left(x \sum_{\substack{d > \sqrt{x} \\ d \in \mathcal{B}_Y}} \frac{1}{d}\right)$$

$$(5.8) \qquad \leq \ C\pi_k(x) \sum_{\substack{Y^Y \leq d \leq \sqrt{x} \\ d \in \mathcal{B}_Y}} \frac{1}{d}\left(\frac{k+c}{x_2}\right)^{\omega(d)} + O\left(\frac{x}{x^{1/4}} \sum_{d \in \mathcal{B}_Y} \frac{1}{\sqrt{d}}\right).$$

Clearly,

$$(5.9) \qquad \sum_{d \in \mathcal{B}_Y} \frac{1}{\sqrt{d}} = \prod_{p \leq Y}\left(1 - \frac{1}{\sqrt{p}}\right)^{-1} < \exp\left\{\sum_{p \leq Y} \frac{1}{\sqrt{p}}\right\} < \exp\{c\sqrt{Y}\}.$$

Thus, it follows from (5.9) that if $E(x)$ stands for the error term in (5.8) and if we choose $Y = Y(x) = x_3$, then we clearly have

$$(5.10) \qquad E(x) \ll x^{4/5},$$

say. On the other hand,

$$\sum_{\substack{Y^Y \leq d \leq \sqrt{x} \\ d \in \mathcal{B}_Y}} \frac{1}{d}\left(\frac{k+c}{x_2}\right)^{\omega(d)} \ \leq \ \frac{1}{Y^{Y/2}} \prod_{p \leq Y}\left(1 + \left(\frac{k+c}{x_2}\right)\left(1 + \frac{1}{\sqrt{p}} + \frac{1}{p} + \ldots\right)\right)$$

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$$(5.11) \qquad = \quad \frac{1}{Y^{Y/2}} \prod_{p \le y} \left( 1 + \frac{(k+c)/x_2}{\sqrt{p}-1} \right) \to 0 \text{ as } Y = Y(x) \to \infty.$$

Hence, using (5.10) and (5.11) in (5.8) and in light of (5.7), we can replace (5.6) by

$$(5.12) \qquad \pi_k(x) = (1 + o(1)) \sum_{\substack{d \in \mathcal{B}_Y \\ d \le Y^Y}} \pi_{k-\omega(d)} \left( \frac{x}{d} | Q_Y \right) \qquad (x \to \infty).$$

Now, consider the two expressions

$$S_k(x) = \sum_{\substack{n \le x \\ n \in \wp_k}} e^{i\tau \nu^*(n)} \qquad \text{and} \qquad S_k(x|D) = \sum_{\substack{n \le x \\ n \in \wp_k \\ (n,D)=1}} e^{i\tau \nu^*(n)}.$$

Then, by choosing $Y = w_x$, we may write

$$S_k(x) = \sum_{\substack{m \in \mathcal{B}_{w_x}}} \sum_{\substack{n \le x/m \\ n \in \wp_{k-\omega(m)} \\ (n,Q_{w_x})=1}} e^{i\tau \nu^*(n)}$$

$$(5.13) \qquad = \sum_{\substack{m \in \mathcal{B}_{w_x} \\ m < w_x^{w_x}}} S_{k-\omega(m)}(\frac{x}{m}|Q) + o(\pi_k(x)) \qquad (x \to \infty).$$

We will now show that the proportion of the non-squarefree integers belonging to the set $\{n \le x : \omega(n) = k, \ p(n) > w_x\}$ is small.

Setting

$$Q := Q_{w_x} \qquad \text{and} \qquad h(n) := \sum_{\substack{p^a \| n \\ a \ge 2}} 1,$$

then we may write

$$(5.14) \qquad \sum_{\substack{n \le x \\ (n,Q)=1, \ n \in \mathcal{N}_k}} h(n) = \sum_{\substack{p^a \le \sqrt{x} \\ a \ge 2 \\ p > Y}} N_{k-a}\left( \frac{x}{p^a} | Q \right) + O\left( x \sum_{\substack{p^a > \sqrt{x} \\ a \ge 2}} \frac{1}{p^a} \right).$$

Now, using Theorem 5 from the book of Tenenbaum ([5], page 205) and using relation (5.1), one can prove that

$$N_k(x|D) = \left( \sum_{\delta | D} \frac{\mu(\delta)}{\delta} \xi^{\Omega(\delta)} \right) N_k(x)(1 + o(1))$$

$$(5.15) \qquad = \prod_{p | D} \left( 1 - \frac{\xi}{p} \right) N_k(x)(1 + o(1)),$$

as $x \to \infty$. Using (5.15) in (5.14), we obtain that

$$\sum_{\substack{n \le x \\ (n,Q)=1, \ n \in \mathcal{N}_k}} h(n) \le cN_k(x|Q) \sum_{\substack{p > w_x \\ \alpha \ge 2}} \frac{1}{p^\alpha} + O(x^{3/4})$$

$$(5.16) \qquad\qquad\qquad \le \varepsilon_1(x)N_k(x|Q),$$

where $\varepsilon_1(x) \to 0$ as $x \to \infty$.

Then, from Lemma 1, we have

$$
\begin{aligned}
\sum_{\substack{n \leq x \\ n \in \wp_k, \ (n,Q)=1}} h(n) &\leq \sum_{\substack{p^\alpha \\ p > Y}} \sum_{\substack{p^\alpha \| n \\ n = p^\alpha m \leq x \\ m \in \wp_{k-1}, \ (m,Q)=1}} 1 \\
&\leq \sum_{\substack{p^\alpha, p > Y \\ \alpha \geq 2}} \pi_{k-1}\left(\frac{x}{p^\alpha}|Q\right) + O(x^{3/4}) \\
&\leq c \prod_{p|Q}\left(1 - \frac{\xi}{p(1-\frac{1-\xi}{p})}\right) \sum_{p^\alpha \leq x} \pi_{k-1}\left(\frac{x}{p^\alpha}\right) + O(x^{3/4}) \\
&\ll \pi_k(x|Q) \sum_{p > Y, \ \alpha \geq 2} \frac{1}{p^\alpha} + O(x^{3/4}) \\
&= \pi_k(x|Q)\varepsilon_3(x) + O(x^{3/4}),
\end{aligned}
$$

(5.17)

where $\varepsilon_3(x) \to 0$ as $x \to \infty$, thus proving our claim that we may ignore those non-squarefree integers for which $\omega(n) = k$ and $p(n) > w_x$.

Hence, from (5.17), we get that

$$
\begin{aligned}
N_k(x|Q) &= \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k, \ (n,Q)=1}} |\mu(n)| + \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k, \ (n,Q)=1 \\ n \ \text{not squarefree}}} 1 \\
&= \sum_{\substack{n \leq x \\ n \in \mathcal{N}_k, \ (n,Q)=1}} |\mu(n)| + O(\varepsilon_3(x)\pi_k(x|Q)),
\end{aligned}
$$

(5.18)

say, where we used (5.17).

We can now move to estimate the main term on the right hand side of (5.13). To do so, we make use of (5.5), which, in light of (5.18), yields. as $x \to \infty$,

$$
\begin{aligned}
S_{k-\omega(m)}\left(\frac{x}{m}|Q\right) &= M_{k-\omega(m)}\left(\frac{x}{m}|Q\right) + o\left(\pi_{k-\omega(m)}\left(\frac{x}{m}|Q\right)\right) \\
&= (1 + o(1))N_{k-\omega(m)}\left(\frac{x}{m}|Q\right)\psi(\tau) \\
&= (1 + o(1))\pi_{k-\omega(m)}\left(\frac{x}{m}|Q\right)\psi(\tau),
\end{aligned}
$$

(5.19)

since $|\psi(\tau)| > c$ for some positive constant $c$ on every finite interval $|\tau| < B$.

Substituting (5.19) in (5.13), we get that, uniformly for $\dfrac{k}{x_2} \in [\delta, 2 - \delta]$,

$$
S_k(x) = (1 + o(1))\pi_k(x)\psi(\tau) \qquad (x \to \infty),
$$

thus completing the proof of Theorem 1.

## 6. Proof of Theorem 2

We will use the method developed in the paper of Bassily, Kátai and Wijsmuller [1].

We first introduce the sequence of completely multiplicative functions $\tau_\ell$, $\ell = 0, 1, \ldots$, which we define on primes $p$ by

$$
\tau_0(p) = 1, \qquad \tau_\ell(p) = \sum_{q|p-1} \tau_{\ell-1}(q) \quad \text{for each } \ell \geq 1.
$$

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From this definition, it is clear that

$$0 \leq \omega(\varphi_\ell(n)) \leq \tau_\ell(n) \qquad \text{for all integers } n \geq 1, \ \ell \geq 0.$$

Note also that Kátai and Subbarao [3] proved that

$$(6.1) \qquad A_\ell(x) \ = \ \sum_{p \leq x} \frac{\tau_\ell(p)}{p} = \frac{1}{(\ell+1)!} x_2^{\ell+1} + O(x_2^\ell),$$

$$(6.2) \qquad B_\ell^2(x) \ = \ \sum_{p \leq x} \frac{\tau_\ell^2(p)}{p} = \frac{x_2^{2\ell+1}}{(2\ell+1)(\ell!)^2} + O\left(\frac{x_2^{2\ell+1/2}}{2\ell+1/2}\right).$$

**Definition.** *We say that the primes $q_0, q_1, \ldots, q_\ell$ constitute an $\ell$-chain if $q_{i-1}|q_i - 1$ for $i = 1, \ldots, \ell$. We denote by $Q_\ell(n)$ those $\ell$-chains such that $q_\ell | n$ and by $Q_\ell(n, q_0)$ those $\ell$-chains with $q_\ell | n$ and starting with $q_0$, in which case we write*

$$q_0 \to q_1 \to \ldots \to q_\ell, \quad q_\ell | n.$$

Given a positive integer $n \in \mathcal{N}_k$ with $n \leq x$, we will now count the number of those $\ell$-chains $q_0 \to q_1 \to \ldots \to q_\ell$, $q_\ell | n$, for which $x^{1/4} < q_\ell < x$. To do so, let us choose $U \in [x^{1/4}, x]$ and let us count those positive integers $n \in \mathcal{N}_k$ for which there is an $\ell$-chain with $q_\ell | n$ and $q_\ell \in [U, 2U]$. For such a $q_\ell$ to exist, we must have $q_0 | q_1 - 1$, $q_1 | q_2 - 1$, $\ldots$, $q_{\ell-1} | q_\ell - 1$, thus implying that any prime $q_\ell$ can be considered at most $\tau_{\ell-1}(q_\ell - 1)$ times. Now, for a given prime $q_\ell$, if $n = q_\ell m \leq x$ with $n \in \mathcal{N}_k$, then we have $m \leq x/U$, $m \in \mathcal{N}_{k-1}$, implying that the number of such $m$'s is at most $cN_{k-1}(x/U)$. We have thus established that the number of such chains is

$$\ll \left\{ \sum_{U < q \leq 2U} \tau_{\ell-1}(q-1) \right\} N_{k-1}\left(\frac{x}{U}\right).$$

Let us now introduce another definition. Let $x$ be a large number and let $\overline{Q_\ell}(n, q_0)$ stand for the set of $\ell$-chains with $q_\ell | n$ which starts at $q_0$ and such that $q_\ell \leq x$. Then, since $|\overline{Q_\ell}(n, q_0)| \geq 1$, we have

$$
\begin{aligned}
L_k^{(1)} \ &:= \ \sum_{n \in \mathcal{N}_k} \sum_{q_0 < y} \left( |\overline{Q_\ell}(n, q_0)| - 1 \right) \\
&\leq \ \sum_{n \in \mathcal{N}_k} \sum_{q_0 < y} |\overline{Q_\ell}(n, q_0)| \\
&\leq \ \sum_{\substack{q_0 \to \ldots \to q_\ell \\ q_\ell < x^{1/4}}} N_{k-1}\left(\frac{x}{q_\ell}\right) \\
&\leq \ cN_{k-1}(x) \sum_{\substack{q_0 \to \ldots \to q_\ell \\ q_\ell < x^{1/4}, \ q_0 < y}} \frac{1}{q_\ell} \\
(6.3) \qquad\qquad &= \ cN_{k-1}(x)E_\ell,
\end{aligned}
$$

say.

Now, using Lemma 2.5 of Bassily, Kátai and Wijsmuller [1], we have that

(6.4) $$E_\ell \leq c \sum_{\substack{q_0 \to \ldots \to q_\ell \\ q_0 < y}} \frac{x_2}{q_{\ell-1}} \leq c E_{\ell-1} x_2 \leq \ldots < c^\ell x_2^\ell E_0,$$

where

$$E_0 = \sum_{q_0 < y} \frac{1}{q_0} < c \log \log y.$$

Substituting (6.4) in (6.3), and since $N_{k-1}(x) \ll_\delta N_k(x)$, we obtain that

(6.5) $$L_k^{(1)} < c_1 x_2^\ell \log \log y \cdot N_k(x).$$

Now, let

$$L_k^{(2)} := \sum_{n \in \mathcal{N}_k} \sum_{q_0 \geq y} \left( |\overline{Q_\ell}(n, q_0)| - 1 \right).$$

Since $|\overline{Q_\ell}(n, q_0)| \neq 1$, it follows that there are at least two chains

$$q_0 \to q_1 \to \ldots \to q_\ell$$
$$q_0' \to q_1' \to \ldots \to q_\ell'$$

such that $q_\ell | n$, $q_\ell' | n$. Using the argument displayed in [1], one can establish that

$$L_k^{(2)} \ll N_k(x) \frac{x_2^{2\ell+1}}{y},$$

so that choosing $y = \log^2 x$, we obtain that

(6.6) $$L_k^{(2)} = o(N_k(x)) \qquad (x \to \infty).$$

It follows from (6.5) and (6.6) that, in order to prove Theorem 2, it is enough to prove it with $\tau_\ell(n)$ instead of $\omega(\varphi_\ell(n))$. Hence we shall prove that, if $\ell \geq 1, a_\ell, b_\ell, \xi = \xi_{k,x}$ are as in Theorem 2 and if we set

$$t_\xi(n) := \frac{\tau_\ell(n) - a_\ell \xi x_2^{\ell+1}}{b_\ell \cdot \sqrt{\xi} \cdot x_2^{\ell+1/2}},$$

then

(6.7) $$\lim_{x \to \infty} \max_{z \in \mathbb{R}} \max_{\xi \in [\delta, 2-\delta]} \left| \frac{1}{N_k(x)} \# \{ n \leq x : n \in \mathcal{N}_k, \ t_\xi(n) < z \} - \Phi(z) \right| = 0.$$

In order to prove relation (3.1) of Theorem 2, we use Theorem 1, while to prove relation (3.2) of Theorem 2, we use the above Theorem A.

We start by choosing the strongly additive function $f$ defined on primes $p$ by $f(p) = \dfrac{\tau_\ell(p) \cdot (\ell+1)!}{x_2^\ell}$. Then, in light of (6.1),

$$A_x = \sum_{p \leq x} \frac{f(p)}{p} = \frac{(\ell+1)!}{x_2^\ell} \sum_{p \leq x} \frac{\tau_\ell(p)}{p} = x_2 + O(1).$$

With the additive function $f^*$ defined on primes $p$ by $f^*(p) = f(p) - \dfrac{A_x}{x_2}$, we have, using (6.2),

$$B_x^2 = \sum_{p \leq x} \frac{f^*(p)^2}{p} = \sum_{p \leq x} \frac{f(p)^2}{p} - 2\frac{A_x}{x_2} \sum_{p \leq x} \frac{f(p)}{p} + \left( \sum_{p \leq x} \frac{1}{p} \right) \frac{A_x^2}{x_2^2}$$

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$$
\begin{aligned}
&= \frac{(\ell+1)!^2}{x_2^{2\ell}} \sum_{p \le x} \frac{\tau_\ell(p)^2}{p} - 2\frac{A_x}{x_2} \sum_{p \le x} \frac{f(p)}{p} + \left(\sum_{p \le x} \frac{1}{p}\right) \frac{A_x^2}{x_2^2} \\
&= \frac{(\ell+1)!^2}{x_2^{2\ell}} \frac{x_2^{2\ell+1}}{(2\ell+1)(\ell!)^2} + O(x_2^{1/2}) - 2\frac{A_x}{x_2} \sum_{p \le x} \frac{f(p)}{p} + \left(\sum_{p \le x} \frac{1}{p}\right) \frac{A_x^2}{x_2^2} \\
&= \frac{(\ell+1)^2}{2\ell+1} x_2 - 2\left(1 + O\left(\frac{1}{x_2}\right)\right)(x_2 + O(1)) + (x_2 + O(1))\left(1 + O\left(\frac{1}{x_2}\right)\right) \\
&= \left(\frac{(\ell+1)^2}{2\ell+1} - 1\right) x_2 + O(\sqrt{x_2}) = \frac{\ell^2}{2\ell+1} x_2 + O(\sqrt{x_2}),
\end{aligned}
$$

thereby satisfying the conditions of Theorem 1 (respectively, Theorem A), thus completing the proof of Theorem 2.

## 7. Further remarks

Using Theorem 1 and Theorem A along with the method elaborated in the paper of Kátai and Subbarao [3], it is possible to deduce theorems of the same type as that of Theorem 2. For instance, it would be possible to prove the following assertion.

**Theorem 3.** *Let $a \ge 1$ and $b \ne 0$ be fixed integers. Consider the multiplicative function $g$ defined on primes $p$ by $g(p) = \max(ap + b, 1)$ and assume that there exists a positive constant $c$ such that $g(p^a) < cp^a$ for all prime powers $p^a$. Assume also that $g(n)$ only takes integer positive values. Further let $g_\ell$ stand for the $\ell$-fold iterate of $g$. Then, there exist computable positive constants $c_\ell$ and $d_\ell$ for which the function*

$$
\mu_\ell(n) := \frac{\omega(g_\ell(n)) - c_\ell \xi x_2^{\ell+1}}{d_\ell \xi x_2^{\ell+1/2}}
$$

*satisfies*

$$
\begin{aligned}
\lim_{x \to \infty} \max_{z \in \mathbb{R}} \max_{\xi \in [\delta, 2-\delta]} \left| \frac{1}{N_k(x)} \#\{n \le x : n \in \mathcal{N}_k,\ \mu_\ell(n) < z\} - \Phi(z) \right| &= 0, \\
\lim_{x \to \infty} \max_{z \in \mathbb{R}} \max_{\xi \in [\delta, 2-\delta]} \left| \frac{1}{\pi_k(x)} \#\{n \le x : n \in \wp_k,\ \mu_\ell(n) < z\} - \Phi(z) \right| &= 0.
\end{aligned}
$$

In particular, Theorem 3 can be applied to the function $g = \sigma$, the sum of the divisors function. It also applies to the multiplicative functions $P$, $P^*$ and $\widetilde{P}$ defined on prime powers $p^a$ by $P(p^a) = (a+1)p^a - ap^{a-1}$, $P^*(p^a) = 2p^a - 1$ and $\widetilde{P}(p^a) = 2p^a - p^{a-1}$, which were introduced and studied by L. Toth [6].

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