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ALGEBRAIC CHARACTERIZATIONS FOR REDUCTION SYSTEMS

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Abstract. We give algebraic characterizations for a reduction system to be respectively Noetherian and confluent, and for a Noetherian reduction system to be confluent. The characterization of a confluent reduction system $(A, \rightarrow)$ is based on a relationship between the confluence of the system and the exactness of the colimit functor colim : $Ab^{\mathcal{A}} \rightarrow Ab$ where $\mathcal{A}$ is the small category with objects the elements of A and arrows $a \rightarrow b$ whenever b is a consequence of a in the system.

1. Introduction

The aim of this paper is to express in algebraic terms the termination and confluence of a reduction system. Three algebraic characterizations for a reduction system $(A, \rightarrow)$ are given, theorems 2 and 3 which give sufficient and necessary conditions for $(A, \rightarrow)$ to be respectively noetherian and confluent, and theorem 5 which gives sufficient and necessary conditions under which a Noetherian reduction system is confluent. For the first two characterizations we use a few results from [8] regarding a unitary ring $[\mathcal{C}]$ which is always associated to a small additive category $\mathcal{C}$. First we explain how the ring is defined and than we mention the results. The underlying set of $[\mathcal{C}]$ is the

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set of $|\mathcal{C}| \times |\mathcal{C}|$ matrices of the form $[\alpha_{p,q}]$ where $\alpha_{p,q} \in \mathcal{C}(p,q)$ and each row and column has finitely many nonzero entries. The addition and multiplication in $[\mathcal{C}]$ are defined by using the addition and composition in $\mathcal{C}$ in the following way

$$[\alpha_{p,q}] + [\beta_{p,q}] = [\alpha_{p,q} + \beta_{p,q}] \text{ and } [\alpha_{p,q}] \cdot [\beta_{p,q}] = [\gamma_{a,b}] \text{ where } \gamma_{a,b} = \sum_{c \in |\mathcal{C}|} \alpha_{a,c} \cdot \beta_{c,b}.$$

It is shown in theorems 7.1 and 7.1* of [8] that the category of right modules $Ab^{[\mathcal{C}]}$ is related to the category of covariant additive functors $Ab^{\mathcal{C}}$ via exact functors

$$Ab^{[\mathcal{C}]} \underset{R,\ S}{\overset{T}{\rightleftarrows}} Ab^{\mathcal{C}} \tag{1}$$

where R and S are respectively right and left adjoint for T. Likewise, for the contravariant case there are adjoint pairs

$$Ab^{[\mathcal{C}]^*} \underset{R^*,\ S^*}{\overset{T^*}{\rightleftarrows}} Ab^{\mathcal{C}^*} \ . \tag{2}$$

As we make use of S^* and R, we recall here briefly that for any $F \in Ab^{\mathcal{C}^*}$, $S^*(F) = \bigoplus_{q \in |\mathcal{C}|} F(q)$ and the action of $\alpha = [\alpha_{p,q}]$ on $S^*(F)$ is given by

$$\alpha u_q = \sum_{p \in |\mathcal{C}|} u_p F(\alpha_{p,q}),$$

where u_q is the coproduct injection. Similarly, $R(F) = \prod_{q \in |\mathcal{C}|} F(q)$ with the action of matrices on the right defined in a similar fashion to the above. Using the above adjunctions, it is shown that for every $F \in Ab^{\mathcal{C}}$ and $G \in Ab^{\mathcal{C}^*}$ there is a natural equivalence

$$SF \otimes_{[\mathcal{C}]} S^*G \simeq F \otimes_{\mathcal{C}} G. \tag{3}$$

For the second characterization, apart from the above results from [8], we use a result of Isbell and Mitchell in [6] which states that categories $\mathcal{C}$ for which the colimit functor colim : $Ab^{\mathcal{C}} \to Ab$ is exact, are precisely those categories whose affinization aff $\mathcal{C}$ has filtered components. Here the affinization of $\mathcal{C}$ is the (nonadditive) subcategory of $\mathbb{Z}\mathcal{C}$ consisting of those morphisms whose integer coefficients sum to 1. In general we have $\mathcal{C} \subseteq$ aff $\mathcal{C}$, with the equality if and only if $\mathcal{C}$ is a preordered set. Being filtered means two things, first any pair of objects map to a common object, and secondly, for any two morphism α_1, α_2 with the same domain and codomain, there exists β such that $\beta\alpha_1 = \beta\alpha_2$. For preordered sets the second condition is superfluous.

A special case of reduction systems are those arising from monoid presentations. If $\mathcal{P} = \langle \mathbf{x}, \mathbf{r} \rangle$ is a presentation of a monoid S, then associated to it there is the reduction system $(\mathbf{x}^*, \to)$ where $\mathbf{x}^*$ is the free monoid on $\mathbf{x}$ and $\to$ is made of pairs $(u\alpha v, u\beta v)$ where $(\alpha, \beta) \in \mathbf{r}$ and $u, v \in \mathbf{x}^*$. In fact this reduction system can be regarded as a disjoint union $\bigsqcup_{s \in S} \langle \mathcal{P}_s, \to_s \rangle$ of reduction systems $\langle \mathcal{P}_s, \to_s \rangle$, where $\mathcal{P}_s$ is the subset of those elements from $\mathbf{x}^*$ which represent the element $s \in S$, and $\to_s$ consists of those pairs of $\to$ with both coordinates inside $\mathcal{P}_s$. If it happens that $\mathcal{P}$ represents a group G, it is interesting to ask if the confluence of $\mathcal{P}_e$ (e is the unit of G) implies that of $\mathcal{P}_g$ for every $g \in G$. We call the presentation $\mathcal{P} = \langle \mathbf{x}, \mathbf{r} \rangle$ of G a λ-confluent presentation whenever the reduction system $\mathcal{P}_e$ is confluent. Here λ is the empty word representing the unit e. We use the result of theorem 3 and proposition 4.1.2, p. 117 of [4] to give a sufficient and necessary condition under which the λ-confluence of a monoid presentation $\mathcal{P} = \langle \mathbf{x}, \mathbf{r} \rangle$ of a group implies the confluence

of $\mathcal{P}$ in the special case when $(\mathcal{P}_e, \to_e)$ is complete, that is confluent and Noetherian. In fact the termination of $\mathcal{P}_e$ implies that of $\mathcal{P}_g$ for if $\tilde{x}$ is a word representing g^{-1}, and if ρ is an infinite sequence of reductions in $\mathcal{P}_g$, then $\tilde{x}\rho$ will be an infinite sequence of reductions in $\mathcal{P}_e$. Proposition 4.1.2 states that if $\varphi : \Lambda \to \Gamma$ is a ring homomorphism, A a right Γ module, C a left Λ module, and if $\mathrm{Tor}_p^{\Lambda}(\Gamma, C) = 0$ for every $p > 0$, then

$$\mathrm{Tor}_n^{\Lambda}(A, C) \simeq \mathrm{Tor}_n^{\Gamma}(A, {}_{(\varphi)}C) \tag{4}$$

where ${}_{(\varphi)}C = \Gamma \otimes_{\Lambda} C$. Note that the isomorphism (4) holds true even in the case when Γ and A, regarded as right Λ modules, are not unitary, therefore we do not need to assume that $\varphi : \Lambda \to \Gamma$ is a ring homomorphism which sends the unit 1_{Λ} of Λ to the unit 1_{Γ} of Γ. Along the proof of proposition 4, we use from [4] (see on page 149) the following definition. If Λ and Γ are two augmented rings with respective augmentations $\varepsilon_{\Lambda} : \Lambda \to Q_{\Lambda}$, $\varepsilon_{\Gamma} : \Gamma \to Q_{\Gamma}$, and $\varphi : \Gamma \to \Lambda$ a map of augmented rings, then there is a map

$$\Psi : \Gamma \otimes_{\Lambda} Q_{\Lambda} \to Q_{\Gamma}$$

defined by $\Psi(\gamma \otimes x) = \gamma \cdot \psi(x)$ where $\psi : Q_{\Lambda} \to Q_{\Gamma}$ is the map induced by φ. Again, the definition of Ψ is still possible under the assumption that Γ is non unitary as a right Λ module via φ.

As we mentioned at the beginning of the introduction, our main objective is to characterize algebraically important notions of the theory of rewriting systems such as confluence and termination proving that such notions are in fact algebraic in nature. Before we explain below the significance of our results, we recall that associated to a reduction system $(A, \to)$ there is the reduction graph Γ_A with vertex set $V(\Gamma_A) = A$ and set of edges

$$E(\Gamma_A) = \{(a, b) : a \in A, b \in A \text{ if there is a reduction rule } a \to b\}.$$

Further, we denote by $F\Gamma_A$ the free category generated by Γ_A, by $\mathbb{Z}F\Gamma_A$ the additive category arising from $F\Gamma_A$ and finally by $[\mathbb{Z}F\Gamma_A]$ the ring associated with $\mathbb{Z}F\Gamma_A$.
Theorem 2 identifies the termination of a reduction system to a chain condition of principal right ideals in a semigroup arising from the system. More specifically, given a finitely branching reduction system $(A, \to)$, that is a system in which every element has finitely many descendants, we denote by RA the multiplicative subsemigroup of the monoid $([\mathbb{Z}F\Gamma_A], \cdot)$ consisting of all those matrices E with finitely many nonzero entries with the additional property that for any $a \in A$, $E_{a,a} \neq z \cdot 1_a$ where 1_a is the identity on a in $F\Gamma_A$ and $z \in \mathbb{Z}$. Our theorem then states that the system is Noetherian if and only if every descending chain of principal right ideals of $(RA, \cdot)$ terminates.
Theorem 3 identifies the confluence of a reduction system $(A, \to)$ with the flatness of a certain module arising from the system. More specifically we let $\mathcal{A}$ be the preorder arising from $F\Gamma_A$ by identifying the parallel arrows and then consider the adjoint situation (2). Theorem 3 then states that $(A, \to)$ is confluent if and only if $S^*\Delta\mathbb{Z}$ is a flat $[\mathbb{Z}\mathcal{A}]^*$ module, where $\Delta\mathbb{Z}$ is the constant functor at $\mathbb{Z}$ over $\mathbb{Z}\mathcal{A}^*$. Proposition 4 is an attempt to get an application of Theorem 3 to the specific situation where the reduction system arises from a monoid presentation $\mathcal{P} = \langle \mathbf{x}, \mathbf{r} \rangle$ of a group G. The problem we try to shade some light on in this proposition, is the so called the problem of λ-confluence which asks under what conditions the confluence of $\mathcal{P}_e$ implies the confluence of $\mathcal{P}_g$ for any $g \in G$. We prove, under the assumption that $(\mathcal{P}_e, \to_e)$ is complete, that for every $g \in G$, $(\mathcal{P}_g, \to_g)$ is confluent if and only if there is an irreducible $v \in \mathcal{P}_g$ such that ${}_{(\varphi_v)}S_e^*\Delta\mathbb{Z} \cong S_g^*\Delta\mathbb{Z}$.
Our final result is Theorem 5 which relates the confluence of a Noetherian reduction system $(A, \to)$ to the algebraic structure of the so called reduction monoid P arising from the system. The monoid P is defined as follows

$$P = \{\tau \in \mathcal{T}(A) : \tau(u) = v \text{ only if } v \text{ is a descendant of } u \text{ or } u = v\},$$

where $\mathcal{T}(A)$ is the full transformation monoid on A, and the operation in P is the usual composition of transformations. Our theorem then states that $(A, \to)$ is complete if and only if the reduction monoid P has a zero element.
We note that the above results connect different areas of pure mathematics, respectively the theory of semigroups and homological algebra to that of rewriting systems. We hope that these connections we have presented here will become helpful in the future in treating problems of rewriting systems.

2. Noetherian reduction systems

Let $(A, \to)$ be a reduction system which is finitely branching. This class of reduction systems is of interest because it includes for example the reduction systems arising from semigroup presentations with finitely many relations.

Lemma 1. *If $(A, \to)$ is a Noetherian and finitely branching reduction system, then for every $a \in A$, there is $n_a \in \mathbb{N}$ such that the length of any path in Γ_A from a to a successor of a does not exceed n_a.*

Proof. Since in such systems every element has finitely many successors, then for every $a \in A$ there are finitely many paths in Γ_A from a to successors of a. If we take n_a to be the maximum of the lengths of those paths, this will do. □

Denote by RA the multiplicative subsemigroup of the monoid $([\mathbb{Z}F\Gamma_A], \cdot)$ consisting of all those matrices E with finitely many nonzero entries with the additional property that for any $a \in A$, $E_{a,a} \neq z \cdot 1_a$ where 1_a is the identity on a in $F\Gamma_A$ and $z \in \mathbb{Z}$.

Theorem 2. *A finitely branching reduction system $(A, \to)$ is Noetherian if and only if every descending chain of principal right ideals of $(RA, \cdot)$ terminates.*

Proof. Let

$$E^{(0)} \cup E^{(0)} \cdot RA \supseteq E^{(1)} \cup E^{(1)} \cdot RA \supseteq \cdots \supseteq E^{(n)} \cup E^{(n)} \cdot RA \supseteq E^{(n+1)} \cup E^{(n+1)} \cdot RA \supseteq \cdots$$

be a descending chain of principal right ideals of RA. For $i \in \mathbb{N}$ we let $Q^{(i)} \in RA$ be such that

$$E^{(i)} = E^{(i-1)} \cdot Q^{(i)}. \tag{5}$$

Using (5), one can show by recursion that

$$E^{(n)}_{a,b} = \sum_{a_n \in A} \cdots \sum_{a_1 \in A} E^{(0)}_{a,a_1} \cdot Q^{(1)}_{a_1,a_2} \cdots Q^{(n-1)}_{a_{n-1},a_n} \cdot Q^{(n)}_{a_n,b}. \tag{6}$$

From the definition of RA, each of the $n+1$ factors of a nonzero term of (6) is made of linear combinations of respectively elements from $\mathrm{Hom}_{F\Gamma_A}(a, a_1)$, $\mathrm{Hom}_{F\Gamma_A}(a_i, a_{i+1})$ and $\mathrm{Hom}_{F\Gamma_A}(a_n, b)$ with integers. If we take one component from each of the above hom-sets that take part in the formation of the terms of (6) we obtain a path in Γ_A of the form

$$a \to a_1 \to a_2 \to \cdots \to a_n \to b \tag{7}$$

whose length is $n+1$ since none of the arrows represented in (7) arises from an identity morphism in $F\Gamma_A$. Let $\mathcal{E}_i$ be the subset of A corresponding to the rows of $E^{(i)}$ which contain nonzero elements. It is easy to see from (5) that $\mathcal{E}_i \subseteq \mathcal{E}_0$. We claim that

$$\forall a \in \mathcal{E}_0, \exists n_a \in \mathbb{N}, \text{ such that } \forall b \in A \text{ with } E^{(n_a)}_{a,b} = 0.$$

Indeed, take n_a as in Lemma 1. Then, path (7) have length $n_a + 1 > n_a$, therefore the term that gave rise (7) could not have been nonzero. Let now $n = \max\{n_a : a \in \mathcal{E}_0\}$. From the above claim we get that $E^{(n)}_{a,b} = 0$ for all $a \in \mathcal{E}_0$ and $b \in A$, therefore $E^{(n)}$ is the zero matrix.

Conversely, suppose that every descending chain of principal right ideals of RA terminates and assume that there exist an infinite path

$$e_1 \cdot e_2 \cdots e_n \cdot e_{n+1} \cdots$$

in Γ_A. Define matrices from RA as follows

$$E^{(n)} = [\varepsilon_{a,b}] \text{ where } \varepsilon_{a,b} = \begin{cases} e_1 \cdots e_n & \text{if} \quad a = \iota(e_1) \text{ and } b = \tau(e_n) \\ 0 & \text{otherwise} \end{cases}$$

It is now obvious that $E^{(n+1)} \subseteq E^{(n)} \cdot RA$ and that $E^{(n)} \notin E^{(n+1)} \cdot RA$ which shows that the descending chain

$$E^{(1)} \cup E^{(1)} \cdot RA \supseteq \cdots \supseteq E^{(n)} \cup E^{(n)} \cdot RA \supseteq E^{(n+1)} \cup E^{(n+1)} \cdot RA \supseteq \cdots$$

does not terminate, a contradiction. □

3. Confluent reduction systems

As before we let $F\Gamma_A$ be the free category generated by the reduction graph associated with the reduction system $(A, \to)$. Let $\mathcal{A}$ be the quotient $F\Gamma_A/\sim$ where $\sim$ is the congruence generated by all the pairs (α, β) with $\alpha, \beta \in F\Gamma_A(a, b)$ and a, b varying in A. In this way $\mathcal{A}$ becomes a preordered set and in this case the elements of $[\mathbb{Z}\mathcal{A}]$ have a simple form: they are $A \times A$ matrices $[\alpha_{p,q}]$ where $\alpha_{p,q} \in \mathbb{Z}$ and each row and column has finitely many nonzero entries. If an entry $\alpha_{p,q}$ is nonzero, then q is a consequence of p in the system $(A, \to)$. Remark here that the confluence of $(A, \to)$ is equivalent to aff $\mathcal{A}$ having filtered components, therefore instead of looking directly for the confluence of $(A, \to)$, one should look for conditions under which aff $\mathcal{A}$ has filtered components. We give such a condition in terms of $[\mathbb{Z}\mathcal{A}]$ and for this purpose we note first that the adjunctions (1) and (2) to the case of the small additive category $\mathbb{Z}\mathcal{A}$ become

$$Ab^{[\mathbb{Z}\mathcal{A}]} \underset{R,\ S}{\overset{T}{\rightleftarrows}} Ab^{\mathbb{Z}\mathcal{A}} \tag{8}$$

and

$$Ab^{[\mathbb{Z}\mathcal{A}]^*} \underset{R^*,\ S^*}{\overset{T^*}{\rightleftarrows}} Ab^{\mathbb{Z}\mathcal{A}^*} \ . \tag{9}$$

Theorem 3. *The reduction system $(A, \to)$ is confluent if and only if $S^*\Delta\mathbb{Z}$ is a flat $[\mathbb{Z}\mathcal{A}]^*$ module, where $\Delta\mathbb{Z}$ is the constant functor at $\mathbb{Z}$ over $\mathbb{Z}\mathcal{A}^*$.*

Proof. For every $G \in Ab^{\mathbb{Z}\mathcal{A}^*}$, every $M \in Ab^{[\mathbb{Z}\mathcal{A}]}$ and every abelian group B, the following natural equivalences hold true

$$\begin{aligned} Ab(S^*G \otimes_{[\mathbb{Z}\mathcal{A}]^*} M, B) &\simeq Ab^{[\mathbb{Z}\mathcal{A}]}(M, Ab(S^*G, B)) \\ &\simeq Ab^{[\mathbb{Z}\mathcal{A}]}(M, RAb(G, B)) \\ &\simeq Ab^{\mathbb{Z}\mathcal{A}}(TM, Ab(G, B)) \\ &\simeq Ab(G \otimes_{\mathbb{Z}\mathcal{A}^*} TM, B). \end{aligned}$$

Then from Yoneda lemma there must be a natural equivalence

$$S^*G \otimes_{[\mathbb{Z}\mathcal{A}]^*} M \simeq G \otimes_{\mathbb{Z}\mathcal{A}^*} TM. \tag{10}$$

If we assume now that $(A, \to)$ is confluent, then aff $\mathcal{A}$ has filtered components and then from [6] the functor $\Delta\mathbb{Z}$ is flat as a right $\mathbb{Z}\mathcal{A}^*$ module. We want to show that $S^*\Delta\mathbb{Z}$ is flat, that is, if $M \to N$ is an injection in $Ab^{[\mathbb{Z}\mathcal{A}]}$, then the induced morphism $S^*\Delta\mathbb{Z}\otimes_{[\mathbb{Z}\mathcal{A}]^*} M \to S^*\Delta\mathbb{Z}\otimes_{[\mathbb{Z}\mathcal{A}]^*} N$ is an injection in Ab. To see this we can use the naturality of (10) by replacing G with $\Delta\mathbb{Z}$ and then obtaining the commutative diagram

$$\begin{array}{ccc} S^*\Delta\mathbb{Z} \otimes_{[\mathbb{Z}\mathcal{A}]^*} M & \longrightarrow & S^*\Delta\mathbb{Z} \otimes_{[\mathbb{Z}\mathcal{A}]^*} N \\ \| & & \| \\ \Delta\mathbb{Z} \otimes_{\mathbb{Z}\mathcal{A}^*} TM & \longrightarrow & \Delta\mathbb{Z} \otimes_{\mathbb{Z}\mathcal{A}^*} TN \end{array}$$

whose vertical arrows are isomorphisms and the bottom arrow is an injection since T preserves injections and $\Delta\mathbb{Z}$ is flat.

Conversely, suppose that $S^*\Delta\mathbb{Z}$ is flat and want to show that for any injection $F_1 \to F_2$ in $Ab^{\mathbb{Z}\mathcal{A}}$, the induced morphism $\Delta\mathbb{Z} \otimes_{\mathbb{Z}\mathcal{A}^*} F_1 \to \Delta\mathbb{Z} \otimes_{\mathbb{Z}\mathcal{A}^*} F_2$ is an injection in Ab. If we apply the natural equivalence (3) to the case when $\mathcal{C} = \mathbb{Z}\mathcal{A}$ and $G = \Delta\mathbb{Z}$, we get the commutative square

$$\begin{array}{ccc} S^*\Delta\mathbb{Z} \otimes_{[\mathbb{Z}\mathcal{A}]^*} SF_1 & \longrightarrow & S^*\Delta\mathbb{Z} \otimes_{[\mathbb{Z}\mathcal{A}]^*} SF_2 \\ \| & & \| \\ \Delta\mathbb{Z} \otimes_{\mathbb{Z}\mathcal{A}^*} F_1 & \longrightarrow & \Delta\mathbb{Z} \otimes_{\mathbb{Z}\mathcal{A}^*} F_2 \end{array}$$

with the vertical arrows being isomorphisms. Since S is an exact functor and $S^*\Delta\mathbb{Z}$ is flat, the top arrow is an injection, therefore the bottom one will be an injection too proving the flatness of $\Delta\mathbb{Z}$. □

4. A discussion on λ-confluence

As we characterized the confluence of a reduction system in terms of the flatness of a certain module arising from the system, then it is natural to use this characterization to obtain information on the confluence of some particular reduction systems. We focus on those reduction systems arising from monoid presentations which give groups with the intention to find conditions under which the corresponding reduction system is confluent.

Let $\mathcal{P} = \langle \mathbf{x}, \mathbf{r} \rangle$ be a monoid presentation for a group G and let $(\mathcal{P}_e, \to_e)$ and $(\mathcal{P}_g, \to_g)$ be the reduction systems corresponding to the unit element e and to some $g \in G$, $g \neq e$. Assume throughout that $(\mathcal{P}_e, \to_e)$ is complete, therefore as we mentioned in the introduction, $(\mathcal{P}_g, \to_g)$ will be terminating. We let $\mathcal{I}_g$ be the set of irreducible words representing g. Denote by Λ the ring $[\mathbb{Z}\,(F\Gamma_{\mathcal{P}_e}/\sim)]$ and by Γ the ring $[\mathbb{Z}\left(F\Gamma_{\mathcal{P}_g}/\sim\right)]$ where $\sim$ is defined as before. Let for any $g \in G$ denote by $R_g^*\Delta\mathbb{Z}$, $S_g^*\Delta\mathbb{Z}$ the modules from Ab^{Γ^*} defined in (9). Define

$$\varepsilon_\Lambda : \Lambda \to R_e^*\Delta\mathbb{Z}$$

by setting for every matrix $\alpha = [\alpha_{p,q}] \in \Lambda$,

$$\pi_p\varepsilon_\Lambda(\alpha) = \sum_{q\in\mathcal{P}_e} \alpha_{p,q}$$

where π_p is the p'th projection. This definition makes sense since α is row finite. It is obviously a group homomorphism and surjective since for every $b \in R_e^*\Delta\mathbb{Z}$ if we take $\alpha \in \Lambda$ such that $\alpha_{p,p} = \pi_p(b)$ for every $p \in \mathcal{P}_e$, and $\alpha_{p,q} = 0$ for $p \neq q$, then from the definition of ε_Λ we see that $\varepsilon_\Lambda(\alpha) = b$. To see that ε_Λ is a homomorphism of left modules we must prove that for every $\alpha, \beta \in \Lambda$ and every $p \in \mathcal{P}_e$, $\pi_p\varepsilon_\Lambda(\alpha \cdot \beta) = \pi_p(\alpha \cdot \varepsilon_\Lambda(\beta))$. Indeed,

$$\pi_p\varepsilon_\Lambda(\alpha \cdot \beta) = \sum_{s\in\mathcal{P}_e}\sum_{q\in\mathcal{P}_e} \alpha_{p,q}\beta_{q,s},$$

and

$$\pi_p(\alpha \cdot \varepsilon_\Lambda(\beta)) = \sum_{q\in\mathcal{P}_e} \alpha_{p,q} \sum_{s\in\mathcal{P}_e} \beta_{q,s},$$

which are equal to each other. We note that the augmentation ideal I_Λ consists in those matrices whose row elements sum to zero. In a similar fashion we define an augmentation homomorphism $\varepsilon_\Gamma : \Gamma \to R_g^*\Delta\mathbb{Z}$ whose augmentation ideal I_Γ again consists in those matrices from Γ whose row elements sum to zero. For every $[\alpha_{p,q}] \in \Lambda$ and every $v \in \mathcal{I}_g$, denote by $v \cdot [\alpha_{p,q}]$ the matrix from Γ whose only nonzero entries are those $\alpha_{vp,vq} = \alpha_{p,q}$ whenever $\alpha_{p,q}$ is nonzero. For each $v \in \mathcal{I}_g$ define

$$\varphi_v : \Lambda \to \Gamma$$

by

$$\varphi_v(\alpha) = v \cdot \alpha.$$

It is easy to see that φ_v is a homomorphism of rings. Also from the definition of φ_v we see that $\varphi_v(I_\Lambda) \subseteq I_\Gamma$ hence it is a map of augmented rings and therefore it induces a Λ module homomorphism $\psi_v : R_e^*\Delta\mathbb{Z} \to R_g^*\Delta\mathbb{Z}$ if we regard $R_g^*\Delta\mathbb{Z}$ as a left Λ module via φ_v. As mentioned in the introduction, ψ_v induces a homomorphism of left Γ modules

$$\Psi_v : {}_{(\varphi_v)}R_e^*\Delta\mathbb{Z} \to R_g^*\Delta\mathbb{Z}$$

defined by

$$\Psi_v(\gamma \otimes a) = \gamma \cdot \psi_v(a).$$

Now using the fact that $S_e^*\Delta\mathbb{Z}$ and $S_g^*\Delta\mathbb{Z}$ are submodules of respectively $R_e^*\Delta\mathbb{Z}$ and $R_g^*\Delta\mathbb{Z}$, and the (easily checking) fact that for every $v \in \mathcal{I}_g$ the only nonzero coordinates of $\psi_v(a)$ are those indexed by vu whenever $\pi_u(a) \neq 0$, one can see that Ψ_v induces a homomorphism of left Γ modules

$$\tilde{\Psi}_v : {}_{(\varphi_v)}S_e^*\Delta\mathbb{Z} \to S_g^*\Delta\mathbb{Z}.$$

Note that for different $v \in \mathcal{I}_g$, modules ${}_{(\varphi_v)}S_e^*\Delta\mathbb{Z}$ may be different. For their coproduct $\bigoplus_{v\in\mathcal{I}_g} {}_{(\varphi_v)}S_e^*\Delta\mathbb{Z}$ we let

$$\tilde{\Psi} : \bigoplus_{v\in\mathcal{I}_g} {}_{(\varphi_v)}S_e^*\Delta\mathbb{Z} \to S_g^*\Delta\mathbb{Z} \tag{11}$$

be the homomorphism of left Γ modules arising from the universal property of coproducts. We show that $\tilde{\Psi}$ is surjective. For this we need only to prove that any generator of the abelian group $S_g^*\Delta\mathbb{Z}$, that is any $d \in S_g^*\Delta\mathbb{Z}$ with $\pi_w(d) = 1$ for a unique $w \in \mathcal{P}_g$, is in the image of $\tilde{\Psi}$. Let $v \in \mathcal{I}_g$ be an irreducible descendant of w. Denote by γ^w the matrix from Γ whose only nonzero entry is $\gamma^w_{w,v} = 1$ and let $c^w \in \bigoplus_{v\in\mathcal{I}_g} {}_{(\varphi_v)}S_e^*\Delta\mathbb{Z}$ such that $\pi_\xi c^w \neq 0$ if and only only if $\xi = v$ and that $\pi_v c^w = \gamma^w \otimes a^w$ where the only nonzero coordinate of a^w is $\pi_\lambda a^w = 1$. The definition of $\tilde{\Psi}$ implies that $\tilde{\Psi}(c^w) = \tilde{\Psi}_v(\gamma^w \otimes a^w) = d$. With the notations established above we prove the following.

Proposition 4. *Let $\mathcal{P} = \langle \mathbf{x}, \mathbf{r} \rangle$ be a monoid presentation for a group G such that the reduction system $(\mathcal{P}_e, \to_e)$ is complete, then for every $g \in G$, $(\mathcal{P}_g, \to_g)$ is confluent if and only if there is $v \in \mathcal{I}_g$ such that ${}_{(\varphi_v)}S_e^*\Delta\mathbb{Z} \cong S_g^*\Delta\mathbb{Z}$.*

Proof. If $(\mathcal{P}_g, \to_g)$ is confluent, then $\mathcal{I}_g$ is a singleton, let say $\mathcal{I}_g = \{v\}$ and in this case (11) has the form

$$\tilde{\Psi} = \Psi_v : {}_{(\varphi_v)}S_e^*\Delta\mathbb{Z} \to S_g^*\Delta\mathbb{Z}.$$

Next we show that $\tilde{\Psi}$ is a split epimorphism of Γ modules. For this we define for each $w \in \mathcal{P}_g$ the family of group homomorphisms

$$\theta_w : \mathbb{Z}_w \to {}_{(\varphi_v)}S_e^*\Delta\mathbb{Z},$$

where $\mathbb{Z}_w$ is an isomorphic copy of the additive group $\mathbb{Z}$, by

$$\theta_w(1) = c^w$$

with $c^w = \gamma^w \otimes a$ where the only nonzero coordinate of a is $\pi_\lambda a = 1$ and the only non zero entry of γ^w is $\gamma^w_{w,v} = 1$. Since $S_g^*\Delta\mathbb{Z}$ as an abelian group is isomorphic to $\bigoplus_{w\in\mathcal{P}_g} \mathbb{Z}_w$, the family θ_w yields an abelian group homomorphism

$$\Theta : S_g^*\Delta\mathbb{Z} \to {}_{(\varphi_v)}S_e^*\Delta\mathbb{Z}$$

such that $\Theta u_w = \theta_w$ where u_w is the coproduct injection. It is easy to see that Θ is a homomorphism of Γ modules. To see that Θ splits, let $d \in S_g^*\Delta\mathbb{Z}$ be any generator with $\pi_w(d) = 1$, then we have

$$\begin{aligned}\tilde{\Psi}\Theta(d) &= \tilde{\Psi}\Theta u_w(1)\\ &= \tilde{\Psi}(\theta_w(1)) = \tilde{\Psi}(c^w) = d.\end{aligned}$$

As a result we get the direct sum of Γ modules ${}_{(\varphi_v)}S_e^*\Delta\mathbb{Z} \cong S_g^*\Delta\mathbb{Z} \oplus K$ where K is the kernel of $\tilde{\Psi}$. But, as we just saw, any generator $\gamma^w \otimes a$ of ${}_{(\varphi_v)}S_e^*\Delta\mathbb{Z}$ is in the image of Θ, therefore Θ is surjective, $K = 0$ and ${}_{(\varphi_v)}S_e^*\Delta\mathbb{Z} \cong S_g^*\Delta\mathbb{Z}$.
For the converse, since $(\mathcal{P}_e, \to_e)$ is confluent, then from theorem 3 $S_e^*\Delta\mathbb{Z}$ is a flat left Λ module, hence if we regard Γ as a right Λ module via φ_v for the given $v \in \mathcal{I}_g$, we have that $\mathrm{Tor}_p^\Lambda(\Gamma, S_e^*\Delta\mathbb{Z}) = 0$ for every $p > 0$, then from proposition 4.1.2 of [4] we obtain the isomorphisms $\mathrm{Tor}_n^\Lambda(A, S_e^*\Delta\mathbb{Z}) \simeq \mathrm{Tor}_n^\Gamma(A, {}_{(\varphi_v)}S_e^*\Delta\mathbb{Z})$ for every $n > 0$ and every right Γ module A. Since $\mathrm{Tor}_n^\Lambda(A, S_e^*\Delta\mathbb{Z}) = 0$, we get the flatness of the left Γ module ${}_{(\varphi_v)}S_e^*\Delta\mathbb{Z}$ hence the flatness $S_g^*\Delta\mathbb{Z}$. Theorem 3 implies that $(\mathcal{P}_g, \to_g)$ is confluent. □

5. Complete reduction systems

In this section we give an algebraic characterization for a Noetherian reduction system to be confluent. Differently from the Newman's lemma ([9]) which states that Noetherian reduction systems are confluent if and only if they are locally confluent, our characterization translates the confluence purely in terms of semigroup theory. First, for every reduction system $(A, \to)$, we construct a submonoid P of the full transformation monoid $\mathcal{T}(A)$ on the set A as follows:

$$P = \{\tau \in \mathcal{T}(A) : \tau(u) = v \text{ only if } v \text{ is a descendant of } u \text{ or } u = v\}.$$

It is clear that, under the usual composition of transformations, P forms a submonoid of $\mathcal{T}(A)$. We call P the reduction monoid of $(A, \to)$. Before we give our characterization, we recall that a Noetherian reduction systems $(A, \to)$ is complete if and only if every element from A has a unique irreducible descendant.

Theorem 5. *Let $(A, \rightarrow)$ be a Noetherian reduction systems. Then, $(A, \rightarrow)$ is complete if and only if the reduction monoid P has a zero element.*

Proof. If $(A, \rightarrow)$ is complete, then, for every $\omega \in A$, the respective congruence class $[\omega]$ has a unique irreducible element, say $irr([\omega])$. Let $\theta \in P$ be the element which sends every $\omega \in A$ to its corresponding $irr([\omega])$. It is easy to show that θ is the zero of P.

Conversely, suppose that P has a zero element θ. Denote by $Irr(\omega)$ the set of irreducibles which are descendants of ω, and write $Irr = \cup_{\omega \in A} Irr(\omega)$. If we think of θ as a $2 \times \infty$ matrix, then we first show that the second row of θ consists only of elements from Irr. Indeed, if there is $u \in A$ such that $\theta(u) = v$ and $v \notin Irr$, then for τ which sends v to some corresponding successor v', we would have $\tau\theta(u) = v'$, which means that $\tau\theta \neq \theta$. Note also that in the second row we always have represented all the elements from Irr because they are not transformed under any element of P. Hence the second row of θ consists only of all the elements of Irr. Next we show that every $\omega \in A$ has a unique irreducible descendant. Suppose by way of contradiction that there is some $u \in A$ which has a set $\{i_\lambda | \lambda \in \Lambda\}$ of distinct irreducible descendants. Let K_λ for $\lambda \in \Lambda$ be respectively $\theta^{-1}(i_\lambda)$. Suppose that $u \in K_\lambda$. Since i_ν with $\nu \neq \lambda$ is a descendant of u too, then there will be some v such that v is a successor of u and i_ν is a descendant of v or $i_\nu = v$. Distinguish between two cases.

(1) $v \notin K_\lambda$. Let $\tau \in P$ be such that it sends u to v. Then $\theta\tau(u) = \theta(v) \neq i_\lambda$ which contradicts the fact that θ is the zero.

(2) $v \in K_\lambda$. Let $\tau \in P$ be such that it sends v to i_ν. Then $\theta\tau(v) = \theta(i_\nu) = i_\nu \neq i_\lambda$ which again contradicts the fact that θ is the zero.

So it remains that u can not have more than one irreducible descendant and hence the system is complete. □

Corollary 6. *A Noetherian reduction system $(A, \rightarrow)$ is complete if and only if the monoid P constructed as above, has cohomological dimension* 0.

Proof. This follows immediately from Theorem 5 and from [5]. □

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