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MAPPINGS AND DECOMPOSITIONS OF PAIRWISE CONTINUITY ON PAIRWISE NEARLY LINDELÖF SPACES

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Abstract. The purpose of this paper is to study the effect of mappings, some decompositions of pairwise continuity and some generalized pairwise open mappings on pairwise nearly Lindelöf spaces. The main result indicates that a pairwise δ-continuous image of a pairwise nearly Lindelöf space is pairwise nearly Lindelöf.

1. Introduction

In literature there are several generalizations of the notion of Lindelöf spaces and these are studied separately for different reasons and purposes. In 1982, Balasubramaniam [1] introduced and studied the notion of nearly Lindelöf spaces. Then in 1996, Cammaroto and Santoro [2] studied and gave further new results about these spaces which are considered as one of the main generalizations of Lindelöf spaces. Recently the authors introduced and studied the notion of pairwise Lindelöf spaces [9] and pairwise nearly Lindelöf spaces [18] and pairwise weakly regular-Lindelf spaces [12] as well as pairwise almost Lindelöf spaces in bitopological setting, see [10] and extended some results due to Balasubramaniam [1] and Cammaroto and Santoro [2].

Our purpose in this paper is to study the decompositions of pairwise continuity concepts, openness and closedness functions and its generalizations concepts, and mappings on pairwise nearly Lindelöf spaces in a suitable way of bitopological spaces after the manner of Fawakhreh and Kılıçman [5]. We extend most of their results in topological spaces to bitopological spaces.

The concepts of continuous functions and its generalizations have been introduced and studied in topological spaces. In [11, 13], the authors studied the pairwise Lindelöfness and pairwise continuity, the authors also introduced and studied the pairwise almost regular-Lindelöf bitopological spaces, their subspaces and subsets, and investigated some of their characterizations (see [14]). In this paper we extend the previous types of continuity to bitopological spaces and investigate their relationships. Moreover, the concepts of open and closed functions and its generalizations also have been introduced and studied in topological spaces. We extend these types of openness and closedness functions to bitopological spaces and investigate their relationship. Some examples and counterexamples will be given in order to establish further relationships.

In section 4, we shall study the effect of mappings, some decompositions of pairwise continuity and some generalized pairwise openness functions on pairwise nearly Lindelöf spaces. We also show that some mappings preserve this property. The main result in our study is that the image of a pairwise nearly Lindelöf space under a pairwise δ-continuous functions is pairwise nearly Lindelöf.

2. Preliminaries

Throughout this paper, all spaces (X,τ) and (X,τ_1,τ_2) (or simply X) are always mean topological spaces and bitopological spaces, respectively. If $\mathcal{P}$ is a topological property, then (τ_i,τ_j)-$\mathcal{P}$ denotes an analogue of this property for τ_i has property $\mathcal{P}$ with respect to τ_j, and p-$\mathcal{P}$ denotes the conjunction (τ_1,τ_2)-$\mathcal{P} \wedge (\tau_2,\tau_1)$-$\mathcal{P}$, i.e., p-$\mathcal{P}$ denotes an absolute bitopological analogue of $\mathcal{P}$. The prefix τ_i-$\mathcal{P}$ denotes the (X,τ_1,τ_2) has a property $\mathcal{P}$ with respect to τ_i. Note that (X,τ_i) has a property $\mathcal{P} \Longleftrightarrow (X,\tau_1,\tau_2)$ has a property τ_i-$\mathcal{P}$.

By τ_i-int(A) and τ_i-cl(A), we shall mean the interior and the closure of a subset A of X with respect to topology τ_i, respectively. By τ_i-open cover of X, we mean that the cover of X by τ_i-open sets in X; similar for the (τ_i,τ_j)-regular open cover of X and etc. The prefixes (τ_i,τ_j)- or τ_i- will be replaced by (i,j)- or i- respectively, if there is no chance for confusion. In this paper always $i,j \in \{1,2\}$ and $i \neq j$.

The concepts of open, regular open, regular closed, preopen and β-open sets are well known in topological spaces. We extend these concepts to bitopological spaces as follows.

Definition 2.1. *A subset S of a bitopological space (X,τ_1,τ_2) is said to be*

(a) *i-open if S is open with respect to τ_i in X, S is called open in X if it is both 1-open and 2-open, or equivalently, $F \in (\tau_1 \cap \tau_2)$ in X;*
(b) *(i,j)-regular open* [8] *if $S = i\text{-int}\,(j\text{-cl}\,(S))$, S is called pairwise regular open if it is both $(1,2)$-regular open and $(2,1)$-regular open;*
(c) *(i,j)-regular closed* [8] *if $S = i\text{-cl}\,(j\text{-int}\,(S))$, S is called pairwise regular closed if it is both $(1,2)$-regular closed and $(2,1)$-regular closed;*
(d) *(i,j)-preopen if $S \subseteq i\text{-int}\,(j\text{-cl}\,(S))$, S is called pairwise preopen if it is both $(1,2)$-preopen and $(2,1)$-preopen;*
(e) *(i,j)-β-open if $S \subseteq j\text{-cl}\,(i\text{-int}\,(j\text{-cl}\,(S)))$, S is called pairwise β-open if it is both $(1,2)$-β-open and $(2,1)$-β-open;*

where $i,j \in \{1,2\}$ and $i \neq j$.

Definition 2.2 (see [6, 9])**.** *A bitopological space (X,τ_1,τ_2) is said to be i-Lindelöf if the topological space (X,τ_i) is Lindelöf. X is called Lindelöf if it is both 1-Lindelöf and 2-Lindelöf. Equivalently, (X,τ_1,τ_2) is Lindelöf if every i-open cover of X has a countable subcover for each $i = 1,2$.*

Definition 2.3 (see [7, 8])**.** *A bitopological space (X,τ_1,τ_2) is said to be (i,j)-regular if for each point $x \in X$ and for each i-open set V containing x, there exists an i-open set U such that $x \in U \subseteq j\text{-cl}\,(U) \subseteq V$. X is called pairwise regular if it is both $(1,2)$-regular and $(2,1)$-regular.*

Definition 2.4 (see [20])**.** *A bitopological space X is said to be (i,j)-almost regular if for each $x \in X$ and for each (i,j)-regular open set V of X containing x, there is* *an (i,j)-regular open set U such that $x \in U \subseteq j\text{-cl}(U) \subseteq V$. The space X is called pairwise almost regular if it is both $(1,2)$-almost regular and $(2,1)$-almost regular.*

Definition 2.5 (see [8, 20])**.** *A bitopological space X is said to be (i,j)-semiregular if for each $x \in X$ and for each i-open set V of X containing x, there is an i-open set U such that $x \in U \subseteq i\text{-int}(j\text{-cl}(U)) \subseteq V$. Similarly, X is called pairwise semiregular if it is both $(1,2)$-semiregular and $(2,1)$-semiregular.*

3. Decompositions of Pairwise Continuity and Pairwise Openness

The concepts of R-map, almost continuous, precontinuous, β-continuous, almost precontinuous, almost β-continuous, δ-continuous and almost δ-continuous functions have been introduced by many authors in a topological space (see [3, 5, 15]). These concepts are extended to bitopological spaces as follows.

Definition 3.1. *A function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is said to be*

(1) *i-continuous if the functions $f : (X,\tau_i) \to (Y,\sigma_i)$ is continuous, f is called continuous if it is i-continuous for each $i = 1,2$;*
(2) *(i,j)-R-map if $f^{-1}(V)$ is (τ_i,τ_j)-regular open set in X for every (σ_i,σ_j)-regular open set V in Y, f is called pairwise R-map if it is both $(1,2)$-R-map and $(2,1)$-R-map*;
(3) *(i,j)-almost continuous if $f^{-1}(V)$ is τ_i-open set in X for every (σ_i,σ_j)-regular open set V in Y, f is called pairwise almost continuous if it is both $(1,2)$-almost continuous and $(2,1)$-almost continuous*;
(4) *(i,j)-precontinuous (resp. (i,j)-β-continuous) if $f^{-1}(V)$ is (τ_i,τ_j)-preopen (resp. (τ_i,τ_j)-β-open) set in X for every σ_i-open set V in Y, f is called pairwise precontinuous (resp. pairwise β-continuous) if it is both $(1,2)$-precontinuous (resp. $(1,2)$-β-continuous) and $(2,1)$-precontinuous (resp. $(2,1)$-β-continuous)*;
(5) *(i,j)-almost precontinuous (resp. (i,j)-almost β-continuous) if for each $x \in X$ and each (σ_i,σ_j)-regular open set V in Y containing $f(x)$, there exists a (τ_i,τ_j)-preopen (resp. (τ_i,τ_j)-β-open) set U in X containing x such that $f(U) \subseteq V$, f is called pairwise almost precontinuous (resp. pairwise almost β-continuous) if it is both $(1,2)$-almost precontinuous (resp. $(1,2)$-almost β-continuous) and $(2,1)$-almost precontinuous (resp. $(2,1)$-almost β-continuous)*;
(6) *(i,j)-δ-continuous (resp. (i,j)-almost δ-continuous) if for each $x \in X$ and each (σ_i,σ_j)-regular open subset V of Y containing $f(x)$, there exists a $((\tau_i,\tau_j))$-regular open subset U of X containing x such that $f(U) \subseteq V$ (resp. $f(U) \subseteq \sigma_j\text{-cl}(V)$), f is called pairwise δ-continuous (resp. pairwise almost δ-continuous) if it is both $(1,2)$-δ-continuous (resp. $(1,2)$-almost δ-continuous) and $(2,1)$-δ-continuous (resp. $(2,1)$-almost δ-continuous).*

Lemma 3.1. *Let $\{A_\alpha : \alpha \in \Delta\}$ be a collection of (i,j)-β-open (resp. (i,j)-preopen) sets in a bitopological space X. Then $\bigcup_{\alpha\in\Delta} A_\alpha$ is (i,j)-β-open (resp. (i,j)-preopen) set in X.*

Proof. We need to prove the (i,j)-β-open part of the lemma. The (i,j)-preopen part can be proved by the similar procedure. For each $\alpha \in \Delta$, since A_α is (i,j)-β-open set in X, we have $A_\alpha \subseteq j\text{-}cl\,(i\text{-}int\,(j\text{-}cl\,(A_\alpha)))$. Then

$$\begin{aligned}
\bigcup_{\alpha\in\Delta} A_\alpha &\subseteq \bigcup_{\alpha\in\Delta} j\text{-cl}\,(i\text{-}int\,(j\text{-}cl\,(A_\alpha))) \\
&\subseteq j\text{-}cl\left(\bigcup_{\alpha\in\Delta} i\text{-}int\,(j\text{-}cl\,(A_\alpha))\right) \\
&\subseteq j\text{-}cl\left(i\text{-}int\left(\bigcup_{\alpha\in\Delta} j\text{-}cl\,(A_\alpha)\right)\right) \\
&\subseteq j\text{-}cl\left(i\text{-}int\left(j\text{-}cl\left(\bigcup_{\alpha\in\Delta} A_\alpha\right)\right)\right).
\end{aligned}$$

Therefore $\bigcup_{\alpha\in\Delta} A_\alpha$ is (i,j)-β-open set in X. □

Theorem 3.1. *The following are equivalent for a function* $f : (X,\tau_1,\tau_2) \to (X,\sigma_1,\sigma_2)$:

(1) f *is* (i,j)*-almost precontinuous* (*resp.* (i,j)*-almost* β*-continuous*);
(2) $f^{-1}(V)$ *is* (τ_i,τ_j)*-preopen* (*resp.* (τ_i,τ_j)*-*β*-open*) *set in* X *for every* (σ_i,σ_j)*-regular open set* V *in* Y.

Proof. (1) $\Longrightarrow$ (2) : Let V be any (σ_i,σ_j)-regular open set in Y and $x \in f^{-1}(V)$. Then $f(x) \in V$, and by (1), there exists a (τ_i,τ_j)-preopen set U_x in X containing x such that $f(U_x) \subseteq V$. Thus $x \in U_x \subseteq f^{-1}(V)$. Therefore, we obtain $f^{-1}(V) = \bigcup_{x\in f^{-1}(V)} U_x$. This shows that $f^{-1}(V)$ is (τ_i,τ_j)-preopen set in X by Lemma 3.1.

(2) $\Longrightarrow$ (1) : Let $x \in X$ and let V be a (σ_i,σ_j)-regular open set in Y containing $f(x)$. Then $x \in f^{-1}(V)$ and by (2), $f^{-1}(V)$ is (τ_i,τ_j)-preopen set in X. So take $U = f^{-1}(V)$, then U is a (τ_i,τ_j)-preopen set in X containing x such that $f(U) = f\left(f^{-1}(V)\right) \subseteq V$. This shows that f is (i,j)-almost continuous.

The proof for the (i,j)-almost β-continuous is similar. □

Corollary 3.1. *The following are equivalent for a function* $f : (X,\tau_1,\tau_2) \to (X,\sigma_1,\sigma_2)$:

(1) f *is pairwise almost precontinuous* (*resp. pairwise almost* β*-continuous*);
(2) $f^{-1}(V)$ *is pairwise preopen* (*resp. pairwise* β*-open*) *set in* X *for every pairwise regular open set* V *in* Y.

Proposition 3.1. *If* $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ *is a pairwise almost continuous function, then* f *is pairwise almost* δ*-continuous.*

Proof. Let $x \in X$ and let V be a (σ_1,σ_2)-regular open set in Y containing $f(x)$. Then $x \in f^{-1}(V)$ and since f is $(1,2)$-almost continuous, $f^{-1}(V)$ is a τ_1-open set in X containing x. Since $W = \tau_1\text{-}int\left(\tau_2\text{-}cl\left(f^{-1}(V)\right)\right)$ is a (τ_1,τ_2)-regular open set in X containing x,

$$f(W) = f\left(\tau_1\text{-}int\left(\tau_2\text{-}cl\left(f^{-1}(V)\right)\right)\right) \subseteq f\left(\tau_2\text{-}cl\left(f^{-1}(V)\right)\right).$$

Since f is also $(2,1)$-almost continuous and $\sigma_2\text{-}cl\,(V)$ is a (σ_2,σ_1)-regular closed set in Y, $\tau_2\text{-}cl\left(f^{-1}(V)\right) \subseteq f^{-1}(\sigma_2\text{-}cl\,(V))$ because $f^{-1}(\sigma_2\text{-}cl\,(V))$ is a τ_2-closed set in X containing $f^{-1}(V)$. So

$$f(W) \subseteq f\left(\tau_2\text{-}cl\left(f^{-1}(V)\right)\right) \subseteq f\left(f^{-1}(\sigma_2\text{-}cl\,(V))\right) \subseteq \sigma_2\text{-}cl\,(V)\,.$$

This shows that f is $(1,2)$-almost δ-continuous. Similarly, f is also $(2,1)$-almost δ-continuous and completes the proof. □

The converse of Proposition 3.1 is not true as the following example shows.

Example 3.1. *Let $X = \{a,b,c\}$ with topologies*

$$\tau_1 = \{\emptyset, \{b\}, \{a,b\}, \{b,c\}, X\},\ \ \tau_2 = \{\emptyset, \{c\}, X\}$$

and

$$\sigma_1 = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\},\ \ \sigma_2 = \{\emptyset, \{a\}, X\}\,.$$

Let $f : (X,\tau_1,\tau_2) \to (X,\sigma_1,\sigma_2)$ be a function defined by $f(a) = b$ and $f(b) = f(c) = c$. Then f is $(1,2)$-almost δ-continuous as well as $(2,1)$-almost δ-continuous so pairwise almost δ-continuous but it is not $(1,2)$-almost continuous since there exists a (σ_1,σ_2)-regular open set $\{b\}$ in (X,σ_1,σ_2) such that $f^{-1}(\{b\}) = \{a\}$ is not τ_1-open set in (X,τ_1,τ_2). Thus f is not pairwise almost continuous. Even f is $(1,2)$-almost δ-continuous but it is not $(1,2)$-δ-continuous since for the (σ_1,σ_2)-regular open set $\{b\}$ in (X,σ_1,σ_2) containing $f(a) = b$, there is no (τ_1,τ_2)-regular open set U in (X,τ_1,τ_2) containing a such that $f(U) \subseteq \{b\}$. It is also not 1-continuous since $f^{-1}(\{b\}) = \{a\}$ is not τ_1-open set in (X,τ_1,τ_2) while $\{b\}$ is σ_1-open set in (X,σ_1,σ_2).

The following we prove that (i,j)-δ-continuity implies (i,j)-almost continuity but the converse is not true as Example 3.2 below shows.

Proposition 3.2. *If $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is (i,j)-δ-continuous function, then f is (i,j)-almost continuous.*

Proof. Let V be a (σ_i,σ_j)-regular open set in Y containing $f(x)$. Since f is (i,j)-δ-continuous function, there exists a (τ_i,τ_j)-regular open set U_x in X containing x such that $f(U_x) \subseteq V$. Then $x \in U_x \subseteq f^{-1}(V)$ and $f^{-1}(V) = \bigcup_{x\in f^{-1}(V)} U_x$. Since every (τ_i,τ_j)-regular open set is τ_i-open, then U_x is τ_i-open set for each x. This implies that $f^{-1}(V)$ is τ_i-open set in X. Therefore f is (i,j)-almost continuous. □

Corollary 3.2. *If $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is pairwise δ-continuous function, then f is pairwise almost continuous.*

Observe that, every i-continuous function is (i,j)-almost continuous and every (i,j)-R-map is (i,j)-almost continuous too, but the converses are not true in general. In fact, i-continuity and (i,j)-R-map property are independent as Example 3.2 and Example 3.3 below show. Moreover, Example 3.2 and Example 3.3 below also shows that i-continuity and (i,j)-δ-continuity are independent concepts. Every (i,j)-R-map is (i,j)-δ-continuous by Lemma 4.1 below but the converse is not true in general as Example 4.1 below show. Furthermore, i-continuity and (i,j)-almost δ-continuity are independent concepts as Example 3.1 above and Example 3.2 below show.

It is also very clear that (i,j)-δ-continuity implies (i,j)-almost δ-continuity but the converse is not true in general as Example 3.1 above shows. The Example 3.1 above and Example 3.2 below show that (i,j)-almost continuity and (i,j)-almost δ-continuity are independent concepts. Furthermore, (i,j)-almost continuity as well as (i,j)-precontinuity implies (i,j)-almost precontinuity, and (i,j)-almost precontinuity as well as (i,j)-β-continuity implies (i,j)-almost β-continuity but the converses are not true in general as Example 3.4, Example 3.5 and Example 3.6 below show. It is very clear that i-continuity implies (i,j)-precontinuity and (i,j)-precontinuity implies (i,j)-β-continuity but the converses are not true as we will see in Example 3.7 and Example 3.8 below.

Example 3.2. *Let $X = \{a, b, c\}$ with topologies*

$$\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}, X\}, \ \tau_2 = \{\emptyset, \{a\}, \{c\}, \{a,c\}, X\}$$

and

$$\sigma_1 = \{\emptyset, \{a\}, \{c\}, \{a,c\}, X\}, \ \sigma_2 = \{\emptyset, \{b,c\}, X\}.$$

Then the function $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ defined by $f(a) = a, f(b) = c$ and $f(c) = b$ is 1-continuous so $(1,2)$-almost continuous. But f is not a $(1,2)$-R-map since $f^{-1}(\{a\}) = \{a\}$ is not (τ_1, τ_2)-regular open set in (X, τ_1, τ_2) while $\{a\}$ is (σ_1, σ_2)-regular open set in (X, σ_1, σ_2). Even f is 1-continuous and $(1,2)$-almost continuous, it is not $(1,2)$-almost δ-continuous since $\{a\}$ is (σ_1, σ_2)-regular open set in (X, σ_1, σ_2) containing $f(a) = a$ but there is no (τ_1, τ_2)-regular open set U in (X, τ_1, τ_2) containing a such that $f(U) \subseteq \sigma_2\text{-}cl\,\{a\} = \{a\}$. Thus f is also not $(1,2)$-δ-continuous.

Example 3.3. *Let $X = \{a, b, c\}$ with topologies*

$$\tau_1 = \{\emptyset, \{c\}, \{a,b\}, X\}, \ \tau_2 = \{\emptyset, \{a\}, \{c\}, \{a,c\}, X\}$$

and

$$\sigma_1 = \{\emptyset, \{a\}, X\}, \ \sigma_2 = \{\emptyset, \{a,c\}, X\}.$$

Then the identity function $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ is a $(1,2)$-R-map since $\emptyset$ and X are the only (σ_1, σ_2)-regular open set in (X, σ_1, σ_2). So f is $(1,2)$-δ-continuous and also $(1,2)$-almost continuous. However f is not 1-continuous since $f^{-1}(\{a\}) = \{a\}$ is not τ_1-open set in (X, τ_1, τ_2) while $\{a\}$ is σ_1-open set in (X, σ_1, σ_2).

Example 3.4. *Let $X = \{a, b, c, d\}$ and $Y = \{x, y, z\}$. Define on X the topologies $\tau_1 = \{\emptyset, \{c\}, \{d\}, \{a,c\}, \{c,d\}, \{a,c,d\}, X\}$, $\tau_2 = \{\emptyset, \{c\}, \{a,c\}, X\}$ and on Y define the topologies $\sigma_1 = \{\emptyset, \{x\}, \{x,y\}, Y\}$, $\sigma_2 = \{\emptyset, Y\}$. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function defined by $f(a) = f(d) = x$, $f(b) = y$ and $f(c) = z$. Then f is a $(1,2)$-R-map thus $(1,2)$-almost continuous, $(1,2)$-almost precontinuous and $(1,2)$-almost β-continuous. But f is not $(1,2)$-β-continuous since $f^{-1}(\{x\}) = \{a,d\}$ is not (τ_1, τ_2)-β-open set in (X, τ_1, τ_2) while $\{x\}$ is σ_1-open set in (Y, σ_1, σ_2). Thus f is neither $(1,2)$-precontinuous nor 1-continuous.*

Example 3.5. *Let $X = \{a, b, c, d\}$ with topologies*

$$\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}, \ \tau_2 = \{\emptyset, \{a,d\}, X\}$$

and

$$\sigma_1 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, X\}, \ \sigma_2 = \{\emptyset, \{a,c\}, X\}.$$

Let $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2)$ be a function defined by $f(a) = a$, $f(b) = f(c) = b$ and $f(d) = d$. Then f is $(1,2)$-almost β-continuous since the (σ_1, σ_2)-regular open *subsets of* (X,σ_1,σ_2) *are* $\emptyset,\{b\}$ *and* X. *But* f *is not* $(1,2)$*-almost precontinuous by Theorem 3.1 since there exists a* (σ_1,σ_2)*-regular open set* $\{b\}$ *in* (X,σ_1,σ_2) *such that* $f^{-1}(\{b\})=\{b,c\}$ *is not* (τ_1,τ_2)*-preopen set in* (X,τ_1,τ_2) *because* $\{b,c\}\nsubseteq \tau_1\text{-}int(\tau_2\text{-}cl(\{b,c\}))=\tau_1\text{-}int(\{b,c\})=\{b\}$.

Example 3.6. *Let* $X=\{a,b,c,d\}$ *with topologies*

$$\tau_1=\{\emptyset,\{c\},\{d\},\{a,c\},\{c,d\},\{a,c,d\},X\},\ \tau_2=\{\emptyset,\{b\},X\}$$

and let $Y=\{x,y,z\}$ *with topologies*

$$\sigma_1=\{\emptyset,\{x\},\{y\},\{x,y\},Y\},\ \sigma_2=\{\emptyset,\{x\},Y\}.$$

Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be a function defined by* $f(a)=z$ *and* $f(b)=f(c)=f(d)=y$. *Then* f *is* $(1,2)$*-almost precontinuous since the* (σ_1,σ_2)*-regular open sets in* (Y,σ_1,σ_2) *are* $\emptyset,\{y\}$ *and* Y. *But* f *is not* $(1,2)$*-almost continuous since there exists a* (σ_1,σ_2)*-regular open set* $\{y\}$ *in* (Y,σ_1,σ_2) *such that* $f^{-1}(\{y\})=\{b,c,d\}$ *is not* τ_1*-open set in* (X,τ_1,τ_2).

Example 3.7. *Let* $X=\{a,b,c\}$ *with topologies*

$$\tau_1=\{\emptyset,\{c\},\{a,b\},X\},\ \tau_2=\{\emptyset,\{c\},X\}$$

and

$$\sigma_1=\{\emptyset,\{a\},X\},\ \sigma_2=\{\emptyset,\{a,c\},X\}.$$

Then the identity function $f:(X,\tau_1,\tau_2)\to(X,\sigma_1,\sigma_2)$ *is* $(1,2)$*-precontinuous. However* f *is not* 1*-continuous since* $f^{-1}(\{a\})=\{a\}$ *is not* τ_1*-open set in* (X,τ_1,τ_2) *while* $\{a\}$ *is* σ_1*-open set in* (X,σ_1,σ_2).

Example 3.8. *Let* $X=\{a,b,c\}$ *with topologies*

$$\tau_1=\{\emptyset,\{a\},\{c\},\{a,c\},X\},\ \tau_2=\{\emptyset,\{c\},X\}$$

and

$$\sigma_1=\{\emptyset,\{a\},\{c\},\{a,c\},\{a,b\},X\},\ \sigma_2=\{\emptyset,\{b\},\{b,c\},X\}.$$

Then the identity function $f:(X,\tau_1,\tau_2)\to(X,\sigma_1,\sigma_2)$ *is* $(1,2)$*-*β*-continuous but it is not* $(1,2)$*-precontinuous since there exists a* σ_1*-open set* $\{a,b\}$ *in* (X,σ_1,σ_2) *such that* $f^{-1}(\{a,b\})=\{a,b\}$ *is not* (τ_1,τ_2)*-preopen set in* (X,τ_1,τ_2) *because* $\{a,b\}\nsubseteq\tau_1\text{-}int(\tau_2\text{-}cl(\{a,b\}))=\tau_1\text{-}int(\{a,b\})=\{a\}$.

From the above discussions, we obtain the following diagram in which none of these implications are reversible.

$$\begin{array}{ccccc}
 & & (i,j)\text{-}R\text{-map} & & \\
 & & \Downarrow & & \\
 & & (i,j)\text{-}\delta\text{-continuous} & \Longrightarrow & (i,j)\text{-almost }\delta\text{-continuous} \\
 & & \Downarrow & & \\
i\text{-continuous} & \Longrightarrow & (i,j)\text{-almost continuous} & & \\
\Downarrow & & \Downarrow & & \\
(i,j)\text{-precontinuous} & \Longrightarrow & (i,j)\text{-almost precontinuous} & & \\
\Downarrow & & \Downarrow & & \\
(i,j)\text{-}\beta\text{-continuous} & \Longrightarrow & (i,j)\text{-almost }\beta\text{-continuous} & &
\end{array}$$

In terms of pairwise properties, we have the following diagram in which none of these implications are reversible. We shall use p- to denote pairwise.

$$p\text{-}R\text{-map}$$

$$\begin{array}{ccccc}
 & & \Downarrow & & \\
 & & p\text{-}\delta\text{-continuous} & & \\
 & & \Downarrow & & \\
\text{continuous} & \Longrightarrow & p\text{-almost continuous} & \Longrightarrow & p\text{-almost } \delta\text{-continuous} \\
\Downarrow & & \Downarrow & & \\
p\text{-precontinuous} & \Longrightarrow & p\text{-almost precontinuous} & & \\
\Downarrow & & \Downarrow & & \\
p\text{-}\beta\text{-continuous} & \Longrightarrow & p\text{-almost } \beta\text{-continuous} & &
\end{array}$$

Many types of open of functions between topological spaces are studied such as almost open, almost α-open, weakly open and M-preopen functions (see [5, 16, 17]). We extend these types of open functions to bitopological setting as follows.

Definition 3.2. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be*

(1) *i-open if the function $f : (X, \tau_i) \to (Y, \sigma_i)$ is open, f is called open if it is both 1-open and 2-open;*
(2) *(i, j)-almost open if $f(U)$ is σ_i-open set in Y for every (τ_i, τ_j)-regular open set U in X, f is called pairwise almost open if it is both $(1, 2)$-almost open and $(2, 1)$-almost open;*
(3) *(i, j)-almost α-open if $f(U) \subseteq \sigma_i\text{-}int\,(\sigma_j\text{-}cl\,(\sigma_i\text{-}int\,(f(U))))$ for every (τ_i, τ_j)-regular open set U in X, f is called pairwise almost α-open if it is both $(1, 2)$-almost α-open and $(2, 1)$-almost α-open;*
(4) *(i, j)-weakly open if $f(U) \subseteq \sigma_i\text{-}int\,(f(\tau_j\text{-}cl\,(U)))$ for every τ_i-open subset U of X, f is called pairwise weakly open if it is both $(1, 2)$-weakly open and $(2, 1)$-weakly open;*
(5) *(i, j)-M-preopen if $f(U)$ is (σ_i, σ_j)-preopen set in Y for every (τ_i, τ_j)-preopen set U in X, f is called pairwise M-preopen if it is both $(1, 2)$-M-preopen and $(2, 1)$-M-preopen.*

The following proposition shows that (i, j)-almost open function implies (i, j)-weakly open.

Proposition 3.3. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-almost open function, then f is (i, j)-weakly open.*

Proof. Let U be a τ_i-open subset of X. Then $\tau_i\text{-}int\,(\tau_j\text{-}cl\,(U))$ is a (τ_i, τ_j)-regular open subset of X. Since f is (i, j)-almost open, then $f(\tau_i\text{-}int\,(\tau_j\text{-}cl\,(U)))$ is a σ_i-open set in Y. Hence $f(\tau_i\text{-}int\,(\tau_j\text{-}cl\,(U))) = \sigma_i\text{-}int\,(f(\tau_i\text{-}int\,(\tau_j\text{-}cl\,(U)))) \subseteq \sigma_i\text{-}int\,(f(\tau_j\text{-}cl\,(U)))$. Since $U \subseteq \tau_i\text{-}int\,(\tau_j\text{-}cl\,(U))$, it implies that

$$f(U) \subseteq f(\tau_i\text{-}int\,(\tau_j\text{-}cl\,(U)))$$

and thus $f(U) \subseteq \sigma_i\text{-}int\,(f(\tau_j\text{-}cl\,(U)))$. This shows that f is (i, j)-weakly open. □

Corollary 3.3. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is pairwise almost open function, then f is pairwise weakly open.*

Observe that every i-open function is (i, j)-almost open but the converse is not true as Example 3.9 below shows. Every (i, j)-almost open function is (i, j)-almost α-open but the converse is not true as Example 3.10 below shows. Although i-openness implies (i, j)-weakly openness but the converse is not true as Example 3.13

below shows. Proposition 3.3 above shows that (i,j)-almost openness implies (i,j)-weakly openness but the converse is not true as Example 3.13 below shows. Moreover, (i,j)-weakly openness and (i,j)-almost α-openness are independent concepts as Example 3.10 and Example 3.13 below show. Furthermore, i-openness and (i,j)-M-preopenness are independent, (i,j)-almost openness and (i,j)-M-preopenness are independent, (i,j)-almost α-openness and (i,j)-M-preopenness are independent, and (i,j)-weakly openness and (i,j)-M-preopenness are also independent concepts by the Example 3.11 and Example 3.12 below show.

Example 3.9. *Let $X = \{a,b,c\}$ with topologies*

$$\tau_1 = \{\emptyset, \{a\}, \{a,b\}, X\}, \ \tau_2 = \{\emptyset, X\}$$

and

$$\sigma_1 = \{\emptyset, \{a\}, X\}, \ \sigma_2 = \{\emptyset, \{b\}, \{b,c\}, X\}.$$

Then the identity function $f : (X,\tau_1,\tau_2) \to (X,\sigma_1,\sigma_2)$ is $(1,2)$-almost open since the only (τ_1,τ_2)-regular open set in (X,τ_1,τ_2) are $\emptyset$ and X. However f is not 1-open since $f(\{a,b\}) = \{a,b\}$ is not σ_1-open set in (X,σ_1,σ_2) for $\{a,b\}$ is τ_1-open set in (X,τ_1,τ_2).

Example 3.10. *Let $X = \{a,b,c,d\}$ with topologies*

$$\begin{aligned} \tau_1 &= \{\emptyset, \{c\}, \{d\}, \{a,c\}, \{c,d\}, \{a,c,d\}, X\}, \\ \tau_2 &= \{\emptyset, \{d\}, \{a,d\}, \{b,d\}, \{a,b,d\}, X\} \end{aligned}$$

and let $Y = \{x,y,z\}$ with topologies

$$\sigma_1 = \{\emptyset, \{z\}, \{x,y\}, Y\}, \ \sigma_2 = \{\emptyset, Y\}.$$

Then a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ defined as $f(a) = f(d) = x$, $f(b) = y$ and $f(c) = z$ is $(1,2)$-almost α-open since the (τ_1,τ_2)-regular open sets in (X,τ_1,τ_2) are $\emptyset, \{c\}, \{a,c\}$ and X. However f is not $(1,2)$-almost open since there exists a (τ_1,τ_2)-regular open set $\{a,c\}$ in (X,τ_1,τ_2) such that $f(\{a,c\}) = \{x,z\}$ is not σ_1-open set in (Y,σ_1,σ_2). Even f is $(1,2)$-almost α-open but it is not $(1,2)$-weakly open since there exists a τ_1-open set $\{a,c\}$ in (X,τ_1,τ_2) such that $f(\{a,c\}) = \{x,z\} \not\subseteq \sigma_1\text{-}int(f(\tau_2\text{-}cl(\{a,c\}))) = \sigma_1\text{-}int(f(\{a,c\})) = \sigma_1\text{-}int(\{x,z\}) = \{z\}$.

Example 3.11. *Let $X = \{a,b,c\}$ with topologies*

$$\tau_1 = \{\emptyset, \{a\}, \{a,b\}, X\}, \ \tau_2 = \{\emptyset, X\}$$

and

$$\sigma_1 = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}, \ \sigma_2 = \{\emptyset, \{b\}, X\}.$$

Then the identity function $f : (X,\tau_1,\tau_2) \to (X,\sigma_1,\sigma_2)$ is 1-open, thus $(1,2)$-almost open, $(1,2)$-almost α-open and $(1,2)$-weakly open. However f is not $(1,2)$-M-preopen since $\{a,c\}$ is a (τ_1,τ_2)-preopen set in (X,τ_1,τ_2) but $f(\{a,c\}) = \{a,c\}$ is not (σ_1,σ_2)-preopen set in (X,σ_1,σ_2) because $f(\{a,c\}) = \{a,c\} \not\subseteq \sigma_1\text{-}int(\sigma_2\text{-}cl(f(\{a,c\}))) = \{a\}$.

Example 3.12. *Let $X = \{a,b,c\}$ with topologies*

$$\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}, \ \tau_2 = \{\emptyset, \{b,c\}, X\}$$

and

$$\sigma_1 = \{\emptyset, \{b\}, X\}, \ \sigma_2 = \{\emptyset, X\}.$$

Then the identity function $f:(X,\tau_1,\tau_2)\to(X,\sigma_1,\sigma_2)$ *is* $(1,2)$*-*M*-preopen since the* $(1,2)$*-preopen sets in* (X,σ_1,σ_2) *are all subsets of* X*. However* f *is neither* $(1,2)$*-weakly open nor* $(1,2)$*-almost* α*-open. For this purpose we take a* τ_1*-open set* $\{a\}$ *in* (X,τ_1,τ_2) *but*

$$f(\{a\})=\{a\}\nsubseteq\sigma_1\text{-}int\,(f(\tau_2\text{-}cl\,(\{a\})))=\sigma_1\text{-}int\,(f(\{a\}))=\sigma_1\text{-}int\,(\{a\})=\emptyset$$

and a (τ_1,τ_2)*-regular open set* $\{a\}$*in* (X,τ_1,τ_2) *but*

$$f(\{a\})=\{a\}\nsubseteq\sigma_1\text{-}int\,(\sigma_2\text{-}cl\,(\sigma_1\text{-}int\,(f(\{a\}))))=\sigma_1\text{-}int\,(\sigma_2\text{-}cl\,(\emptyset))=\emptyset$$

in (X,σ_1,σ_2)*. Thus* f *is also neither* $(1,2)$*-almost open nor* 1*-open by direct implications.*

Example 3.13. *Let* $X=\{a,b,c\}$ *with* $\tau_1=\{\emptyset,\{a\},\{c\},\{a,c\},X\}$, $\tau_2=\{\emptyset,\{c\},X\}$ *and let* $Y=\{x,y\}$ *with* $\sigma_1=\{\emptyset,Y\}$, $\sigma_2=\{\emptyset,\{x\},Y\}$*. Let* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be a function defined by* $f(a)=f(c)=x$ *and* $f(b)=y$*. Then* f *is* $(1,2)$*-weakly open but it is not* $(1,2)$*-almost* α*-open since there exists a* (τ_1,τ_2)*-regular open set* $\{a\}$ *in* (X,τ_1,τ_2) *such that*

$$f(\{a\})=\{x\}\nsubseteq\sigma_1\text{-}int\,(\sigma_2\text{-}cl\,(\sigma_1\text{-}int\,(f(\{a\}))))=\sigma_1\text{-}int\,(\sigma_2\text{-}cl\,(\emptyset))=\emptyset$$

in (Y,σ_1,σ_2)*. Thus* f *is neither* $(1,2)$*-almost open nor* 1*-open by direct implication or by* $f(\{a\})=\{x\}$ *is not* σ_1*-open set in* (Y,σ_1,σ_2) *for* $\{a\}$ *is a* (τ_1,τ_2)*-regular open set or a* τ_1*-open set in* (X,τ_1,τ_2)*.*

Therefore, we obtain the following diagram in which none of these implications are reversible.

$$\begin{array}{ccccc} i\text{-open} & \Longrightarrow & (i,j)\text{-almost open} & \Longrightarrow & (i,j)\text{-almost }\alpha\text{-open} \\ & & \Downarrow & & \\ & & (i,j)\text{-weakly open} & & \end{array}$$

In terms of pairwise properties, we have the following diagram in which none of these implications are reversible.

$$\begin{array}{ccccc} p\text{-open} & \Longrightarrow & p\text{-almost open} & \Longrightarrow & p\text{-almost }\alpha\text{-open} \\ & & \Downarrow & & \\ & & p\text{-weakly open} & & \end{array}$$

4. Mapping on Pairwise Nearly Lindelöf Spaces

Definition 4.1 (see [18])**.** *A bitopological space* (X,τ_1,τ_2) *is said to be* (τ_i,τ_j)*-nearly Lindelöf if for every* τ_i*-open cover* $\{U_\alpha:\alpha\in\Delta\}$ *of* X*, there exists a countable subset* $\{\alpha_n:n\in\mathbb{N}\}$ *of* Δ *such that* $X=\bigcup_{n\in\mathbb{N}}\tau_i\text{-}int\,(\tau_j\text{-}cl\,(U_{\alpha_n}))$*, or as is easily seen to be equivalent, if every* (τ_i,τ_j)*-regular open cover of* X *has a countable subcover.* X *is called pairwise nearly Lindelöf if it is both* (τ_1,τ_2)*-nearly Lindelöf and* (τ_2,τ_1)*-nearly Lindelöf.*

It is also equivalent to say that, a bitopological space X is (i,j)-nearly Lindelöf if and only if every (i,j)-regular open cover of X has a countable subcover (see[18]).

It is well known that in a topological space and a bitopological space, the continuous image of a Lindelöf space is Lindelöf. While Fawakhreh and Kılıçman [5] stated that the δ-continuous image of a nearly Lindelöf space is nearly Lindelöf. For the (τ_i, τ_j)-nearly Lindelöf spaces we give the following theorem.

Theorem 4.1. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a surjective and (i, j)-δ-continuous function. If X is (τ_i, τ_j)-nearly Lindelöf, then Y is (σ_i, σ_j)-nearly Lindelöf.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a (σ_i, σ_j)-regular open cover of Y. Let $x \in X$ and let $\alpha_x \in \Delta$ such that $f(x) \in V_{\alpha_x}$. Since f is (i, j)-δ-continuous, there exists a (τ_i, τ_j)-regular open set U_{α_x} of X containing x such that $f(U_{\alpha_x}) \subseteq V_{\alpha_x}$. So $\{U_{\alpha_x} : x \in X\}$ forms a (τ_i, τ_j)-regular open cover of X. Since X is (τ_i, τ_j)-nearly Lindelöf, there exists a countable subset $\{x_n : n \in \mathbb{N}\}$ of X such that $X = \bigcup_{n\in\mathbb{N}} U_{\alpha_{x_n}}$. Since f is surjective, we have $Y = f(X) = f\left(\bigcup_{n\in\mathbb{N}} U_{\alpha_{x_n}}\right) = \bigcup_{n\in\mathbb{N}} f\left(U_{\alpha_{x_n}}\right) \subseteq \bigcup_{n\in\mathbb{N}} V_{\alpha_{x_n}}$ which implies $Y = \bigcup_{n\in\mathbb{N}} V_{\alpha_{x_n}}$. This shows that Y is (σ_i, σ_j)-nearly Lindelöf and completes the proof. □

Corollary 4.1. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a surjective and pairwise δ-continuous function. If X is pairwise nearly Lindelöf, then so is Y.*

Lemma 4.1. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an (i, j)-R-map, then f is (i, j)-δ-continuous.*

Proof. Let $x \in X$ and let V be a (σ_i, σ_j)-regular open subset of Y containing $f(x)$. Then $x \in f^{-1}(V)$. Since f is an (i, j)-R-map, $f^{-1}(V)$ is a (τ_i, τ_j)-regular open set in X. So if $U = f^{-1}(V)$, then U is a (τ_i, τ_j)-regular open subset of X containing x such that $f(U) = f\left(f^{-1}(V)\right) \subseteq V$. This shows that f is (i, j)-δ-continuous. □

Corollary 4.2. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise R-map, then f is pairwise δ-continuous.*

The converse of Lemma 4.1 is not true as the following example shows.

Example 4.1. *Let $X = \{a, b, c, d\}$ with topologies*

$$\begin{aligned}\tau_1 &= \{\emptyset, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\},\\ \tau_2 &= \{\emptyset, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}, X\}\end{aligned}$$

and $Y = \{x, y, z\}$ with topologies

$$\sigma_1 = \{\emptyset, \{x\}, \{y\}, \{x, y\}, Y\}, \ \sigma_2 = \{\emptyset, \{x, z\}, Y\}.$$

Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function defined by $f(a) = z$, $f(b) = f(c) = f(d) = y$. Then f is $(1, 2)$-δ-continuous but it is not $(1, 2)$-R-map since there exists a (σ_1, σ_2)-regular open set $\{y\}$ in (Y, σ_1, σ_2) such that $f^{-1}(\{y\}) = \{b, c, d\}$ is not (τ_1, τ_2)-regular open set in (X, τ_1, τ_2).

By using Lemma 4.1 and Theorem 4.1 above, we have the following corollary.

Corollary 4.3. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a surjective and (i, j)-R-map. If X is (τ_i, τ_j)-nearly Lindelöf, then Y is (σ_i, σ_j)-nearly Lindelöf.*

Corollary 4.4. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a surjective and pairwise R-map. If X is pairwise nearly Lindelöf, then Y is pairwise nearly Lindelöf.*

Lemma 4.2. *Every pairwise almost continuous and (i,j)-almost α-open function is an (i,j)-R-map.*

Proof. Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a pairwise almost continuous and (i,j)-almost α-open function. Let V be a (σ_i,σ_j)-regular open set in Y. Since f is (i,j)-almost continuous, $f^{-1}(V)$ is a τ_i-open set in X. So $f^{-1}(V)\subseteq\tau_i$-$int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)$. Next we have to show the opposite inclusion. Since f is (i,j)-almost α-open and τ_i-$int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)$ is a (τ_i,τ_j)-regular open set in X, we have

$$\begin{aligned}&f\left(\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\right)\\ \subseteq\ &\sigma_i\text{-}int\left(\sigma_j\text{-}cl\left(\sigma_i\text{-}int\left(f\left(\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\right)\right)\right)\right)\\ \subseteq\ &\sigma_i\text{-}int\left(\sigma_j\text{-}cl\left(\sigma_i\text{-}int\left(f\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\right)\right)\right).\end{aligned}$$

Since f is (j,i)-almost continuous and σ_j-$cl(V)$ is a $\sigma_j\sigma_i$-regular closed set in Y, τ_j-$cl\left(f^{-1}(V)\right)\subseteq f^{-1}(\sigma_j\text{-}cl(V))$ because $f^{-1}(\sigma_j\text{-}cl(V))$ is a τ_j-closed set in X containing $f^{-1}(V)$. So

$$\begin{aligned}f\left(\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\right)&\subseteq\sigma_i\text{-}int\left(\sigma_j\text{-}cl\left(\sigma_i\text{-}int\left(f\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\right)\right)\right)\\&\subseteq\sigma_i\text{-}int\left(\sigma_j\text{-}cl\left(\sigma_i\text{-}int\left(f\left(f^{-1}(\sigma_j\text{-}cl(V))\right)\right)\right)\right)\\&\subseteq\sigma_i\text{-}int(\sigma_j\text{-}cl(\sigma_i\text{-}int(\sigma_j\text{-}cl(V))))\\&\subseteq\sigma_i\text{-}int(\sigma_j\text{-}cl(V))=V.\end{aligned}$$

Thus

$$\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\subseteq f^{-1}\left(f\left(\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\right)\right)\subseteq f^{-1}(V).$$

Hence $f^{-1}(V)=\tau_i$-$int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)$ which implies that $f^{-1}(V)$ is a (τ_i,τ_j)-regular open set in X. This shows that f is an (i,j)-R-map and completes the proof. □

Corollary 4.5. *Every pairwise almost continuous and pairwise almost α-open function is a pairwise R-map.*

Corollary 4.6. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a surjective, pairwise almost continuous and (i,j)-almost α-open function. If X is (τ_i,τ_j)-nearly Lindelöf, then Y is (σ_i,σ_j)-nearly Lindelöf.*

Proof. It is a direct consequence of Lemma 4.2 and Corollary 4.3 above. □

Corollary 4.7. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a surjective, pairwise almost continuous and pairwise almost α-open function. If X is pairwise nearly Lindelöf, then so is Y.*

Since (i,j)-almost open function is (i,j)-almost α-open, by using Lemma 4.2 we obtain the following lemma.

Lemma 4.3. *Every pairwise almost continuous and (i,j)-almost open function is an (i,j)-R-map.*

Corollary 4.8. *Every pairwise almost continuous and pairwise almost open function is a pairwise R-map.*

Corollary 4.9. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a surjective, pairwise almost continuous and (i,j)-almost open function. If X is (τ_i,τ_j)-nearly Lindelöf, then Y is (σ_i,σ_j)-nearly Lindelöf.*

Proof. It is a direct consequence of Lemma 4.3 and Corollary 4.3 above. It is also a direct consequence of Corollary 4.6 above. □

Corollary 4.10. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a surjective, pairwise almost continuous and pairwise almost open function. If X is pairwise nearly Lindelöf, then so is Y.*

Lemma 4.4. *Every pairwise almost continuous and (i,j)-weakly open function is an (i,j)-R-map.*

Proof. Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a pairwise almost continuous and (i,j)-weakly open function. Let V be a (σ_i,σ_j)-regular open set in Y. Since f is (i,j)-almost continuous, $f^{-1}(V)$ is a τ_i-open set in X. So $f^{-1}(V)\subseteq\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)$. Next we have to show the opposite inclusion. Since f is (i,j)-weakly open and τ_i-$int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)$ is also τ_i-open set in X, we have

$$\begin{aligned} f\left(\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\right) &\subseteq \sigma_i\text{-}int\left(f\left(\tau_j\text{-}cl\left(\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\right)\right)\right)\\ &\subseteq \sigma_i\text{-}int\left(f\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\right). \end{aligned}$$

Since f is (j,i)-almost continuous, $\tau_j\text{-}cl\left(f^{-1}(V)\right)\subseteq f^{-1}\left(\sigma_j\text{-}cl(V)\right)$. So

$$\begin{aligned} f\left(\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\right) &\subseteq \sigma_i\text{-}int\left(f\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\right)\\ &\subseteq \sigma_i\text{-}int\left(f\left(f^{-1}\left(\sigma_j\text{-}cl(V)\right)\right)\right)\\ &\subseteq \sigma_i\text{-}int\left(\sigma_j\text{-}cl(V)\right)=V. \end{aligned}$$

Thus

$$\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\subseteq f^{-1}\left(f\left(\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\right)\right)\subseteq f^{-1}(V).$$

Hence $f^{-1}(V)=\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)$ which implies that $f^{-1}(V)$ is a (τ_i,τ_j)-regular open set in X. This shows that f is an (i,j)-R-map and completes the proof. □

Corollary 4.11. *Every pairwise almost continuous and pairwise weakly open function is a pairwise R-map.*

By using Lemma 4.4 and Corollary 4.3 above, we conclude the following corollary.

Corollary 4.12. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a surjective, pairwise almost continuous and (i,j)-weakly open function. If X is (τ_i,τ_j)-nearly Lindelöf, then Y is (σ_i,σ_j)-nearly Lindelöf.*

Corollary 4.13. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a surjective, pairwise almost continuous and pairwise weakly open function. If X is pairwise nearly Lindelöf, then so is Y.*

Lemma 4.5. *Every pairwise almost continuous and (i,j)-M-preopen function is an (i,j)-R-map.*

Proof. Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a pairwise almost continuous and (i,j)-M-preopen function. Let V be a (σ_i,σ_j)-regular open set in Y. Since f is (i,j)-almost continuous, $f^{-1}(V)$ is a τ_i-open set in X. So $f^{-1}(V)\subseteq\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)$. Next we have to show the opposite inclusion. Since f is (i,j)-M-preopen and

$$\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\subseteq\tau_i\text{-}int\left(\tau_j\text{-}cl\left(\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\right)\right),$$

i.e., τ_i-$int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)$ is a (τ_i,τ_j)-preopen set in X,

$$f\left(\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\right)$$

is a (σ_i,σ_j)-preopen set in Y, i.e.,

$$\begin{aligned} f\left(\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\right) &\subseteq \sigma_i\text{-}int\left(\sigma_j\text{-}cl\left(f\left(\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\right)\right)\right) \\ &\subseteq \sigma_i\text{-}int\left(\sigma_j\text{-}cl\left(f\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\right)\right). \end{aligned}$$

Since f is (j,i)-almost continuous, $\tau_j\text{-}cl\left(f^{-1}(V)\right) \subseteq f^{-1}(\sigma_j\text{-}cl(V))$. So

$$\begin{aligned} f\left(\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\right) &\subseteq \sigma_i\text{-}int\left(\sigma_j\text{-}cl\left(f\left(f^{-1}(\sigma_j\text{-}cl(V))\right)\right)\right) \\ &\subseteq \sigma_i\text{-}int(\sigma_j\text{-}cl(\sigma_j\text{-}cl(V))) \\ &= \sigma_i\text{-}int(\sigma_j\text{-}cl(V)) = V. \end{aligned}$$

Thus

$$\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right) \subseteq f^{-1}\left(f\left(\tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)\right)\right) \subseteq f^{-1}(V).$$

Hence $f^{-1}(V) = \tau_i\text{-}int\left(\tau_j\text{-}cl\left(f^{-1}(V)\right)\right)$ which implies that $f^{-1}(V)$ is a (τ_i,τ_j)-regular open set in X. This shows that f is an (i,j)-R-map and completes the proof. □

Corollary 4.14. *Every pairwise almost continuous and pairwise M-preopen function is a pairwise R-map.*

By using Lemma 4.5 and Corollary 4.3, we have the following corollary.

Corollary 4.15. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a surjective, pairwise almost continuous and (i,j)-M-preopen function. If X is (τ_i,τ_j)-nearly Lindelöf, then Y is (σ_i,σ_j)-nearly Lindelöf.*

Corollary 4.16. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a surjective, pairwise almost continuous and pairwise M-preopen function. If X is pairwise nearly Lindelöf, then so is Y.*

Let X be a topological space. A cover $\mathcal{V}=\{V_\lambda:\lambda\in\Lambda\}$ of X is a refinement [2, 4] of another cover $\mathcal{U}=\{U_\alpha:\alpha\in\Delta\}$ if for each $\lambda\in\Lambda$, there exists an $\alpha(\lambda)\in\Delta$ such that $V_\lambda\subseteq U_{\alpha(\lambda)}$, i.e., each $V\in\mathcal{V}$ is contained in some $U\in\mathcal{U}$. If the elements of $\mathcal{V}$ are open sets, we will call $\mathcal{V}$ an open refinement of $\mathcal{U}$; if they are closed sets, we call $\mathcal{V}$ a closed refinement. A family $\mathcal{U}=\{U_\alpha:\alpha\in\Delta\}$ of subsets of a topological space X is locally finite [2, 4] if for every point $x\in X$, there exists a neighbourhood U_x of x such that the set $\{\alpha\in\Delta:U_x\cap U_\alpha\neq\emptyset\}$ is finite, i.e., each $x\in X$ has a neighbourhood U_x meeting only finitely many $U\in\mathcal{U}$.

If bitopological space (X,τ_1,τ_2) considered, i-locally finite concept appear as follows.

Definition 4.2. *A family $\mathcal{U}=\{U_\alpha:\alpha\in\Delta\}$ of subsets of a space (X,τ_1,τ_2) is i-locally finite if for every point $x\in X$, there exists an i-neighbourhood U_x of x such that the set $\{\alpha\in\Delta:U_x\cap U_\alpha\neq\emptyset\}$ is finite, i.e., each $x\in X$ has an i-neighbourhood U_x meeting only finitely many $U\in\mathcal{U}$.*

In 1969, Singal and Arya [19] introduced the notion of nearly paracompact spaces in topological spaces. Now we extend this notion to bitopological setting as follows.

Definition 4.3. *A bitopological space X is said to be (i,j)-nearly paracompact if every cover of X by (i,j)-regular open sets admits an i-locally finite refinement (not necessarily 1-open or 2-open). X is called pairwise nearly paracompact if it is both $(1,2)$-nearly paracompact and $(2,1)$-nearly paracompact.*

Cammaroto and Santoro [2] proved that an almost regular and nearly Lindelöf space is nearly paracompact. We extend this result to bitopological setting as follows.

Lemma 4.6. *Let (X,τ_1,τ_2) be an (i,j)-almost regular and (i,j)-nearly Lindelöf space. Then X is (i,j)-nearly paracompact.*

Proof. Let $\mathcal{V} = \{V_\alpha : \alpha \in \Delta\}$ be an (i,j)-regular open cover of X. For each $x \in X$, there exists $\alpha_x \in \Delta$ such that $x \in V_{\alpha_x}$. Since X is (i,j)-almost regular, there exists an (i,j)-regular open neighbourhood U_{α_x} of x such that $x \in U_{\alpha_x} \subseteq j\text{-}cl\,(U_{\alpha_x}) \subseteq V_{\alpha_x}$. So $\{U_{\alpha_x} : x \in X\}$ is an (i,j)-regular open cover of X. Since X is (i,j)-nearly Lindelöf, there exists a countable subset of points $x_1, x_2, \ldots, x_n, \ldots$ of X such that $X = \bigcup_{n\in\mathbb{N}} U_{\alpha_{x_n}}$. For each $n \in \mathbb{N}$, put $G_n = V_{\alpha_{x_n}} \setminus \left(\bigcup_{k=1}^{n-1} j\text{-}cl\left(U_{\alpha_{x_k}}\right)\right)$. By construction $\{G_n : n \in \mathbb{N}\}$ is an i-locally finite family. In fact, if $x \in X$ then there exist $U_{\alpha_{x_p}}$ (since $\left\{U_{\alpha_{x_n}} : n \in \mathbb{N}\right\}$ is a cover of X) and $V_{\alpha_{x_p}}$ such that $x \in U_{\alpha_{x_p}} \subseteq V_{\alpha_{x_p}}$. We will prove that $U_{\alpha_{x_p}}$ intersects at most finitely many members of the family $\{G_n : n \in \mathbb{N}\}$. Since $G_1 = V_{\alpha_{x_1}}, G_2 = V_{\alpha_{x_2}} \setminus j\text{-}cl\,(U_{\alpha_{x_1}}), \ldots, G_p = V_{\alpha_{x_p}} \setminus \left(j\text{-}cl\,(U_{\alpha_{x_1}}) \cup \cdots \cup j\text{-}cl\left(U_{\alpha_{x_{p-1}}}\right)\right), G_{p+1} = V_{\alpha_{x_{p+1}}} \setminus \left(j\text{-}cl\,(U_{\alpha_{x_1}}) \cup \cdots \cup j\text{-}cl\left(U_{\alpha_{x_p}}\right)\right)$, $U_{\alpha_{x_p}} \cap G_r = \emptyset$ for each $r \geq p+1$. Therefore $U_{\alpha_{x_p}}$ intersects at most a finite number of sets in the family $\{G_n : n \in \mathbb{N}\}$. Next we assert that $\{G_n : n \in \mathbb{N}\}$ is the required refinement of $\mathcal{V}$. Let x be any point of X. We wish to prove that x lies in an element of $\{G_n : n \in \mathbb{N}\}$. Consider the cover $\left\{V_{\alpha_{x_n}} : n \in \mathbb{N}\right\}$ of X; let N be the smallest integer such that x lies in $V_{\alpha_{x_N}}$. Observe that the point x is not lies in G_k for $k < N$ but x lies in G_N since it is not lies in $\bigcup_{k=1}^{N-1} j\text{-}cl\left(U_{\alpha_{x_k}}\right)$. Therefore $x \in \bigcup_{n\in\mathbb{N}} G_n$ which implies that $\{G_n : n \in \mathbb{N}\}$ covers X. This shows that X is (i,j)-nearly paracompact. □

Corollary 4.17. *Let (X,τ_1,τ_2) be a pairwise almost regular and pairwise nearly Lindelöf space. Then X is pairwise nearly paracompact.*

Note that, if (X,τ_1,τ_2) is (i,j)-semiregular and (i,j)-nearly Lindelöf then it is i-Lindelöf (see [14]). Thus by this fact and Lemma 4.6, we conclude the following corollaries.

Corollary 4.18. *Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be a surjective function satisfying one of the following conditions:*

(1) *(i,j)-δ-continuous,*
(2) *(i,j)-R-map,*
(3) *pairwise almost continuous and (i,j)-almost α-open,*
(4) *pairwise almost continuous and (i,j)-almost open,*
(5) *pairwise almost continuous and (i,j)-weakly open,*

(6) *pairwise almost continuous and* (i,j)*-*M*-preopen.*
If X *is* (τ_i,τ_j)*-nearly Lindelöf and* Y *is a* (σ_i,σ_j)*-semiregular (resp.* (σ_i,σ_j) *-almost regular) space, then* Y *is* σ_i*-Lindelöf (resp.* (σ_i,σ_j) *-nearly paracompact).*

Corollary 4.19. *Let* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be a surjective function satisfying one of the following conditions:*
(1) *pairwise* δ*-continuous,*
(2) *pairwise* R*-map,*
(3) *pairwise almost continuous and pairwise almost* α*-open,*
(4) *pairwise almost continuous and pairwise almost open,*
(5) *pairwise almost continuous and pairwise weakly open,*
(6) *pairwise almost continuous and pairwise* M*-preopen.*
If X *is pairwise nearly Lindelöf and* Y *is a pairwise semiregular (resp. pairwise almost regular) space, then* Y *is Lindelöf (resp. pairwise nearly paracompact).*

Since an (i,j)-regular space is (i,j)-semiregular and (i,j)-almost regular (see [14]), we have the following corollary.

Corollary 4.20. *Let* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be a surjective function satisfying one of the conditions* $(1)-(6)$ *of Corollary 4.18. If* X *is* (τ_i,τ_j)*-nearly Lindelöf and* Y *is a* (σ_i,σ_j)*-regular space, then* Y *is* σ_i*-Lindelöf and* (σ_i,σ_j)*-nearly paracompact.*

Corollary 4.21. *Let* $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ *be a surjective function satisfying one of the conditions* $(1)-(6)$ *of Corollary 4.19. If* X *is pairwise nearly Lindelöf and* Y *is a pairwise regular space, then* Y *is Lindelöf and pairwise nearly paracompact.*

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