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ITERATIVE METHODS FOR GENERAL NONCONVEX VARIATIONAL INEQUALITIES

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Abstract. In this paper, we introduce and consider some new classes of variational inequalities and the Wiener-Hopf equations. Using the projection technique, we establish the equivalence between the general nonconvex variational inequalities and the fixed point problems as well as the Wiener-Hopf equations. This alternative equivalent formulation is used to study the existence of a solution of the general convex variational inequalities. This equivalence is used to suggest and analyzed several projection iterative methods for solving the general nonconvex variational inequalities. Convergence criteria of these new iterative is also discussed under suitable conditions. Our method of proofs is very simple as compared with other techniques.

1. Introduction

Variational inequalities theory, which was introduced in early sixties, has emerged as an interesting and fascinating field of mathematical and engineering sciences. It is tool of great power that can be applied to a wide variety of problems, which arise in almost all branches of pure, applied, physical, regional and engineering sciences. It have been shown that the variational inequalities provide the most natural, direct, simple and efficient framework for the general treatment of wide range of problems, see [1-35] and the references therein.

In recent years, variational inequalities have been generalized in several directions using novel and innovative techniques. Noor [25,26,28] has introduced and considered some classes of variational inequalities in the setting of uniformly prox-regular sets. It is known [6,7,33] that the uniformly prox-regular sets are nonconvex and include the convex sets as special cases. Inspired and motivated by ongoing research in this direction, we introduce and consider a new class of general nonconvex variational inequalities involving two (nonlinear) operators. This work is continuation of our earlier work. Using the idea and technique of Noor [25,26,28], we show that the projection technique can be extended for the general nonconvex variational inequalities. We establish the equivalence between the general nonconvex variational inequalities and fixed point problems using essentially the projection technique. This equivalent alternative formulation is used to discuss the existence of a solution of the nonconvex variational inequalities, which is Theorem 3.1. We use this alternative equivalent formulation to suggest and analyze an implicit type

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iterative methods for solving the nonconvex variational inequalities. In order to implement this new implicit method, we use the predictor-corrector technique to suggest a two-step method for solving the nonconvex variational inequalities, which is Algorithm 3.4. We also consider the convergence (Theorem 3.2) of the new iterative method under some suitable conditions. We have also suggested three-step iterative methods for solving nonconvex variational inequalities. Some special cases are also discussed.

We also introduce and consider the problem of solving the nonlinear Wiener-Hopf equations. Using essentially the projection technique, we establish the equivalence between the general nonconvex variational inequalities and the Wiener-Hopf equations. This alternative equivalent formulation is more general and flexible than the projection operator technique. This alternative equivalent formulation is used to suggest and analyze a number of iterative methods for solving the nonconvex variational inequalities. These iterative methods is the subject of Section 4. We also consider the convergence criteria of the proposed iterative methods under some suitable conditions. Several special cases are also discussed. Results obtained in this paper can be viewed as refinement and improvement of the previously known results for the variational inequalities and related optimization problems. We would like to point out that our method of proofs is very simple as compared with other techniques.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let K be a nonempty and convex set in H.

We, first of all, recall the following well-known concepts from nonlinear convex analysis and nonsmooth analysis [7,33].

Definition 2.1. The proximal normal cone of K at $u \in H$ is given by

$$N_K^P(u) := \{\xi \in H : u \in P_K[u + \alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in K : d_K(u) = \|u - u^*\|\}.$$

Here $d_K(.)$ is the usual distance function to the subset K, that is

$$d_K(u) = \inf_{v \in K} \|v - u\|.$$

The proximal normal cone $N_K^P(u)$ has the following characterization.

Lemma 2.1. Let K be a nonempty, closed and convex subset in H. Then $\zeta \in N_K^P(u)$,
if and only if, there exists a constant $\alpha > 0$ such that

$$\langle \zeta, v - u \rangle \leq \alpha \|v - u\|^2, \quad \forall v \in K.$$

Definition 2.2. The Clarke normal cone, denoted by $N_K^C(u)$, is defined as

$$N_K^C(u) = \overline{co}[N_K^P(u)],$$

where $\overline{co}$ means the closure of the convex hull. Clearly $N_K^P(u) \subset N_K^C(u)$, but the converse is not true. Note that $N_K^P(u)$ is always closed and convex, whereas $N_K^C(u)$ is convex, but may not be closed (Ref. 24).

Poliquin et al. [33] and Clarke et al [7] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions.

Definition 2.3. For a given $r \in (0, \infty]$, a subset K_r is said to be normalized uniformly r-prox-regular if and only if every nonzero proximal normal to K_r can be realized by an r-ball, that is, $\forall u \in K_r$ and $0 \neq \xi \in N^P_{K_r}(u)$, one has

$$\langle (\xi)/\|\xi\|, v - u\rangle \leq (1/2r)\|v - u\|^2, \quad \forall v \in K.$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [7,33]. It is clear that if $r = \infty$, then uniformly prox-regularity of K_r is equivalent to the convexity of K. It is known that if K_r is a uniformly prox-regular set, then the proximal normal cone $N^P_{K_r}(u)$ is closed as a set-valued mapping. Thus, we have $N^P_{K_r}(u) = N^C_{K_r}(u)$.

For a given nonlinear operator T, h, we consider the problem of finding $u \in K_r$ such that

$$\langle \rho Tu + u - h(u), h(v) - u\rangle \geq 0, \quad \forall v \in H : h(v) \in K_r, \tag{1}$$

which is called the *general nonconvex variational inequality.*

If $h \equiv I$, the identity operator, then problem (1) is equivalent to finding $u \in K_r$ such that

$$\langle \rho Tu, v - u\rangle \geq 0, \quad \forall v \in K_r, \tag{2}$$

which is known as the nonconvex variational inequality, studied and introduced by Noor [26].

We note that, if $K_r \equiv K$, the convex set in H, then problem (1) is equivalent to finding $u \in K$ such that

$$\langle \rho Tu + u - h(u), h(v) - u\rangle \geq 0, \qquad \forall v \in H : h(u) \in K. \tag{3}$$

Inequality of type (3) is called the *general variational inequality*, which was introduced and studied by Noor [28].

If $h(u) = u$, then problem (1) is equivalent to finding $u \in H : h(u) \in K_r$ such that

$$\langle T(h(u)), h(v) - h(u)\rangle \geq 0, \quad \forall v \in H : h(v) \in K_r, \tag{4}$$

which is also called the general nonconvex variational inequality.

If $K_r \equiv K$, the convex set in H, then problem (4) is equivalent to finding $u \in H : h(u) \in K$ such that

$$\langle T(h(u)), h(v) - h(u)\langle \geq 0, \forall v \in H; h(v) \in K, \tag{5}$$

which was introduced and studied by Noor [12] in 1988. It has been shown that the minimum of a differentiable nonconvex function can be characterized by the general variational inequality (5). See also [19] for its applications in applied sciences.

If $h \equiv I$, the identity operator, then problem (5) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad v \in K, \tag{6}$$

which is known as the classical variational inequality, introduced and studied by Stampacchia [35] in 1964. It turned out that a number of unrelated obstacle, free, moving, unilateral and equilibrium problems arising in various branches of pure and applied sciences can be studied via variational inequalities, see [1-35] and the references therein.

It is well-known that problem (6) is equivalent to finding $u \in K$ such that

$$0 \in Tu + N_K(u), \tag{7}$$

where $N_K(u)$ denotes the normal cone of K at u in the sense of convex analysis. Problem (7) is called the variational inclusion associated with variational inequality (6).

Similarly, if K_r is a nonconvex (uniformly prox-regular) set, then problem (1) is equivalent to finding $u \in K_r$ such that

$$0 \in \rho Tu - h(u) + u + \rho N^P_{K_r}(u), \tag{8}$$

where $N^P_{K_r}(u)$ denotes the normal cone of K_r at u in the sense of nonconvex analysis. Problem (8) is called the nonconvex variational inclusion problem associated with nonconvex variational inequality (1). This implies that the general nonconvex variational inequality (1) is equivalent to finding a zero of the sum of two monotone operators (8). This equivalent formulation plays a crucial and basic part in this paper. We would like to point out this equivalent formulation allows us to use the projection operator technique for solving the general nonconvex variational inequality (1).

We now recall the well known proposition which summarizes some important properties of the uniform prox-regular sets.

Lemma 2.2. Let K be a nonempty closed subset of H, $r \in (0, \infty]$ and set $K_r = \{u \in H : d(u, K) < r\}$. If K_r is uniformly prox-regular, then

i. $\forall u \in K_r, P_{K_r}(u) \neq \emptyset$.

ii. $\forall r^{'} \in (0, r)$, P_{K_r} is Lipschitz continuous with constant $\frac{r}{r-r'}$ on $K_{r'}$.

iii. The proximal normal cone is closed as a set-valued mapping.

We now consider the problem of solving the nonlinear Wiener-Hopf equations. To be more precise, let $Q_{K_r} = I - hP_{K_r}$, where P_{K_r} is the projection operator, h is the nonlinear operator and I is the identity operator. For given nonlinear operators T, h, consider the problem of finding $z \in H$ such that

$$TP_{K_r}z + \rho^{-1}Q_{K_r}z = 0. \tag{9}$$

Equations of the type (9) are called the general nonconvex Wiener-Hopf equations. Note that, if $r = \infty$ and $h =\equiv I$, the identity operator, then the nonlinear Wiener-Hopf equations are exactly the same Wiener-Hopf equations associated with the variational inequalities (6), which were introduced and studied by Shi [34]. This shows that the original Wiener-Hopf equations are the special case of the nonlinear Wiener-Hopf equations (9). The Wiener-Hopf equations technique has been used to study and develop several iterative methods for solving variational inequalities and related optimization problems, see [9-26].

Definition 2.4. An operator $T : H \to H$ is said to be:

(i) *strongly monotone*, if and only if, there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \alpha||u - v||^2, \quad \forall u, v \in H.$$

(ii) *Lipschitz continuous*, if and only if, there exists a constant $\beta > 0$ such that

$$||Tu - Tv|| \leq \beta||u - v||, \quad \forall u, v \in H.$$

3. Projection Methods

In this section, we establish the equivalence between the nonconvex variational inequality (1) and the fixed point problem using the projection operator technique. This alternative formulation is used to discuss the existence of a solution of the problem (1) and to suggest some new iterative methods for solving the general nonconvex variational inequality (1).

Lemma 3.1. $u \in K_r$ is a solution of the general nonconvex variational inequality (1) if and only if $u \in K_r$ satisfies the relation

$$u = P_{K_r}[h(u) - \rho Tu], \tag{10}$$

where P_{K_r} is the projection of H onto the uniformly prox-regular set K_r.

Proof. Let $u \in K_r$ be a solution of (1). Then, for a constant $\rho > 0$,

$$\begin{aligned} 0 &\in u + \rho N^P_{K_r}(u) - (h(u) - \rho Tu) = (I + \rho N^P_{K_r})(u) - (h(u) - \rho Tu) \\ &\Longleftrightarrow \\ u &= (I + \rho N^P_{K_r})^{-1}[h(u) - \rho Tu] = P_{K_r}[h(u) - \rho Tu], \end{aligned}$$

where we have used the well-known fact that $P_{K_r} \equiv (I + N^P_{K_r})^{-1}$. □

Lemma 3.1 implies that the general nonconvex variational inequality (1) is equivalent to the fixed point problem (10). This alternative equivalent formulation is very useful from the numerical and theoretical point of views.

We rewrite the the relation (10) in the following form

$$F(u) = P_{K_r}[h(u) - \rho Tu], \tag{11}$$

which is used to study the existence of a solution of the general nonconvex variational inequality (1).

We now study those conditions under which the general nonconvex variational inequality (1) has a solution and this is the main motivation of our next result.

Theorem 3.1. Let P_{K_r} be the Lipschitz continuous operator with constant $\delta = \frac{r}{r-r'}$. Let T, h be strongly monotone with constants $\alpha > 0, \sigma > 0$ and Lipschitz continuous with constants $\beta > 0, \delta > 0$, respectively. If there exists a constant $\rho > 0$ such that

$$\begin{aligned} |\rho - \frac{\alpha}{\beta^2}| &< \frac{\sqrt{\delta^2\alpha^2 - \beta^2(\delta^2 - (1-\delta k)^2))}}{\delta\beta^2}, \\ \delta\alpha &> \beta\sqrt{\delta^2 - (1-\delta k)^2}, \quad < \delta(1+k), \end{aligned} \tag{12}$$

then there exists a solution of the problem (1).

Proof. From Lemma 3.1, it follows that problems (10) and (1) are equivalent. Thus it is enough to show that the map $F(u)$, defined by (11), has a fixed point. For all $u \neq v \in K_r$, we have

$$\begin{aligned} ||F(u) - F(v)|| &= \|P_{K_r}[h(u) - \rho Tu] - P_{K_r}[h(v) - \rho Tv]|| \\ &\le \delta\|h(u) - h(v) - \rho(Tu - Tv)\| \\ &\le \delta\left\{\|u - v - (h(u) - h(v)) + \|u - v - \rho(Tu - Tv)\|\right\}, \end{aligned} \tag{13}$$

where we have used the fact that the operator P_{K_r} is a Lipschitz continuous operator with constant δ.

Since the operator T is strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, it follows that

$$\begin{aligned} ||u - v - \rho(Tu - Tv)||^2 &\leq ||u - v||^2 - 2\rho\langle Tu - Tv, u - v\rangle + \rho^2||Tu - Tv||^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)||u - v||^2. \end{aligned} \tag{14}$$

In a similar way, we have

$$\|u - v - (h(u) - h(v))\| \leq \sqrt{1 - 2\sigma + \delta^2}\|u - v|, \tag{15}$$

where $\sigma > 0$ is the strongly monotonicity constant and $\delta > 0$ is the Lipschitz continuity constant of the operator h respectively.

From (13), (14) and (15), we have

$$\begin{aligned} ||F(u) - F(v)|| &\leq \delta\left\{k + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2)}\right\}||u - v|| \\ &= \theta||u - v||, \end{aligned}$$

where

$$\theta = \delta\{\sqrt{1 - 2\alpha\rho + \beta^2\rho^2} + k\} \tag{16}$$

$$k = \sqrt{1 - 2\sigma + \delta^2}. \tag{17}$$

From (12), it follows that $\theta < 1$, which implies that the map $F(u)$ defined by (11), has a fixed point, which is the unique solution of (1). $\square$

This fixed point formulation (10) is used to suggest the following iterative method for solving the nonconvex variational inequality (1).

Algorithm 3.1. For a given $u_0 \in K_r$, find the approximate solution u_{n+1} by the iterative scheme

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n\{P_{K_r}[h(u_n) - \rho Tu_n]\}, \quad n = 0, 1, 2, \ldots, \tag{18}$$

where $\alpha_n \in [0, 1], \forall n \geq 0$ is a constant. Algorithm 3.1 is also called the Mann iteration process.

For $\alpha_n = 1$, Algorithm 3.1 collapse to:

Algorithm 3.2. For a given $u_0 \in K_r$, find the approximate solution u_{n+1} by the iterative scheme

$$h(u_{n+1}) = P_{K_r}[h(u_n) - \rho Tu_n], \quad n = 0, 1, 2, \ldots$$

We again use the fixed formulation to suggest and analyze an iterative method for solving the nonconvex variational inequalities (1) as:

Algorithm 3.3. For a given $u_0 \in K_r$, find the approximate solution u_{n+1} by the iterative scheme

$$h(u_{n+1}) = P_{K_r}[h(u_{n+1}) - \rho Tu_{n+1}], \quad n = 0, 1, 2, \ldots$$

Algorithm 3.3 is an implicit type iterative method, which is difficult to implement. To implement Algorithm 3.3, we use the predictor-corrector technique. Here we use the Algorithm 3.1 as a predictor and Algorithm 3.3 as a corrector. Consequently, we have the following iterative method

Algorithm 3.4. For a given $u_0 \in K_r$, find the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} y_n &= P_{K_r}[h(u_n) - \rho Tu_n] \\ u_{n+1} &= P_{K_r}[h(y_n) - \rho Ty_n], \quad n = 0, 1, 2, \ldots \end{aligned}$$

which is called the two-step or splitting type iterative method for solving the general nonconvex variational inequalities (1). It is worth mentioning that Algorithm 3.4 can be suggested by using the updating the technique of the solution.

In this paper, we suggest and analyze the following two-step iterative method for solving the general nonconvex variational inequalities (1).

Algorithm 3.5. For a given $u_0 \in K_r$, find the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} y_n &= (1-\beta_n)u_n + \beta_n\{P_{K_r}[h(u_n) - \rho T u_n]\} \\ u_{n+1} &= (1-\alpha_n)u_n + \alpha_n\{P_{K_r}[h(y_n) - \rho T y_n]\}, \quad n = 0, 1, 2, \ldots, \end{aligned}$$

where $\alpha_n, \beta_n \in [0,1], \quad \forall n \geq 0.$

Clearly for $\alpha_n = \beta_n = 1$, Algorithm 3.5 reduces to Algorithm 3.4. It is worth mentioning that, if $r = \infty$, then the nonconvex set K_r reduces to a convex set K. Consequently Algorithms 3.1- 3.5 collapse to the following algorithms for solving the classical variational inequalities (6). We would like to point that Algorithm 3.4 appears to be a new one for solving the variational inequalities (3)

We now consider the convergence analysis of Algorithm 3.1 and this is the main motivation of our next result. In a similar way, one can consider the convergence criteria of other Algorithms.

Theorem 3.2. Let P_{K_r} be the Lipschitz continuous operator with constant $\delta = \frac{r}{r-r'}$. Let the operators $T, h : H \longrightarrow H$ be strongly monotone with constants $\alpha > 0, \sigma > 0$ and Lipschitz continuous with constants with $\beta > 0, \delta > 0$, respectively. If (12) holds, $\alpha_n, \in [0,1], \quad \forall n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the approximate solution u_n obtained from Algorithm 3.1 converges to a solution $u \in K_r$ satisfying the nonconvex variational inequality (1).

Proof. Let $u \in K_r$ be a solution of the nonconvex variational inequality (1). Then, using Lemma 3.1, we have

$$u = (1-\alpha_n)u + \alpha_n\{P_{K_r}[h(u) - \rho T u]\}, \tag{19}$$

where $0 \leq \alpha_n \leq 1$ is a constant.

From (14)-(19) and using the Lipschitz continuity of the projection P_{K_r} with constant δ, we have

$$\begin{aligned} ||u_{n+1} - u|| &= ||(1-\alpha_n)(u_n - u) + \alpha_n\{P_{K_r}[h(u_n) - \rho T u_n] - P_{K_r}[h(u) - \rho T u]\}|| \\ &\leq (1-\alpha_n)\|u_n - u\| + \alpha_n\|P_{K_r}[h(u_n) - \rho T u_n] - P_{K_r}[h(u) - \rho T u]\| \\ &\leq (1-\alpha_n)\|u_n - u\| + \alpha_n\delta\left\{\|u_n - u + \rho(Tu_n - Tu)\|\right\} \\ &\quad + \alpha_n\|u_n - u - (h(u_n) - h(u))\| \\ &\leq (1-\alpha_n)\|u_n - u\| + \alpha_n\delta\{k + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\}\|u_n - u\| \\ &= [1 - \alpha_n(1-\theta)]\,\|u_n - u\| \\ &\leq \prod_{i=0}^{n}[1 - \alpha_i(1-\theta)]\,\|u_0 - u\|, \end{aligned}$$

where, using (12), we have

$$\theta = \delta\left\{k + \delta\sqrt{1 - 2\rho\alpha + \beta^2\rho^2}\right\} < 1.$$

Since $\sum_{n=0}^{\infty} \alpha_n$ diverges and $1-\theta > 0$, we have $\lim_{n\to\infty} \{\prod_{i=0}^{n}[1-(1-\theta)\alpha_i]\} = 0$. Consequently the sequence $\{u_n\}$ convergences strongly to u. This completes the proof. □

4. Wiener-Hopf Equations Technique

In this section, we first establish the equivalence between the nonconvex variational inequality (1) and the Wiener-Hopf equations (9) using essentially the projection method. This equivalence is used to suggest and analyze some iterative methods for solving the variational inclusions.

Using Lemma 3.1, we show that the general nonconvex variational inequality (1) are equivalent to the Wiener-Hopf equations (9).

Lemma 4.1. The nonconvex variational inequality (1) has a solution $u \in K_r$ if and only if the Wiener-Hopf equations (9) have a solution $z \in H$, provided

$$u = P_{K_r} z \tag{20}$$

$$z = h(u) - \rho T u, \tag{21}$$

where $\rho > 0$ is a constant.

Proof. Let $u \in K_r$ be a solution of (1). Then, from Lemma 3.1, we have

$$u = P_{K_r}[h(u) - \rho T u]. \tag{22}$$

Taking $z = h(u) - \rho T u$ in (22), we have

$$u = P_{K_r} z. \tag{23}$$

From (22) and (23), we have

$$z = h(u) - \rho T u = h P_{K_r} z - \rho T P_{K_r} z,$$

which shows that $z \in H$ is a solution of the Wiener-Hopf equations (9). This completes the proof. □

From Lemma 4.1, we conclude that the general nonconvex variational inequality (1) and the Wiener-Hopf equations (9) are equivalent. This alternative formulation plays an important and crucial part in suggesting and analyzing various iterative methods for solving variational inequalities and related optimization problems. In this paper, by suitable and appropriate rearrangement, we suggest a number of new iterative methods for solving the general nonconvex variational inequality (1).

I. The Wiener-Hopf equations (9) can be written as

$$Q_{K_r} z = -\rho T P_{K_r} z,$$

which implies that, using(4.2)

$$z = h P_{K_r} z - \rho T P_{K_r} z = h(u) - \rho T u.$$

This fixed point formulation enables us to suggest the following iterative method for solving the nonconvex variational inequality (1).

Algorithm 4.1. For a given $z_0 \in H$, compute z_{n+1} by the iterative schemes

$$u_n = P_{K_r} z_n \tag{24}$$

$$z_{n+1} = (1-\alpha_n) z_n + \alpha_n \{h(u_n) - \rho T u_n,\} \quad n = 0, 1, 2, \ldots\ldots, \tag{25}$$

where $0 \le \alpha_n \le 1$, for all $n \ge 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

II. The Wiener-Hopf equations (9) may be written as

$$\begin{aligned} z &= hP_{K_r}z - \rho TP_{K_r}z + (1-\rho^{-1})Q_{K_r}z \\ &= h(u) - \rho Tu + (1-\rho^{-1})Q_{K_r}z. \end{aligned}$$

Using this fixed point formulation, we suggest the following iterative method.

Algorithm 4.2. For a given $z_0 \in H$, compute z_{n+1} by the iterative schemes

$$\begin{aligned} u_n &= P_{K_r}z_n \\ z_{n+1} &= (1-\alpha_n)z_n + \alpha_n\{h(u_n) - \rho Tu_n + (1-\rho^{-1})Q_{K_r}z_n,\} \quad n = 0,1,2,\dots, \end{aligned}$$

where $0 \leq \alpha_n \leq 1$, for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

III. If the operator T is linear and T^{-1} exists, then the Wiener-Hopf equations (9) can be written as

$$z = (I - \rho^{-1}T^{-1})Q_{K_r}z,$$

which allows us to suggest the iterative method.

Algorithm 4.3. For a given $z_0 \in H$, compute z_{n+1} by the iterative scheme

$$z_{n+1} = (1-\alpha_n)z_n + \alpha_n\{(I - \rho^{-1}T^{-1})Q_{K_r}z_n,\} \quad n = 0,1,2\dots,$$

where $0 \leq \alpha_n \leq 1$, for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

We would like to point out that one can obtain a number of iterative methods for solving the general nonconvex variational inequality (1) for suitable and appropriate choices of the operators T, h and the space H. This shows that iterative methods suggested in this paper are more general and unifying ones.

We now study the convergence analysis of Algorithm 4.1. In a similar way, one can analyze the convergence analysis of other iterative methods.

Theorem 4.1. Let the operators T, A satisfy all the assumptions of Theorem 3.1. If the condition (12) holds, $\alpha_n \in [0,1]$, $\forall n \geq 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the approximate solution $\{z_n\}$ obtained from Algorithm 4.1 converges to a solution $z \in H$ satisfying the Wiener-Hopf equation (9) strongly.

Proof. Let $u \in H$ be a solution of (1). Then, using Lemma 4.1, we have

$$z = (1-\alpha_n)z + \alpha_n\{h(u) - \rho Tu\}, \tag{26}$$

where $0 \leq \alpha_n \leq 1$, and $\sum_{n=0}^{\infty} a_n = \infty$.

From(25), (26), (14) and (15), we have

$$\begin{aligned} \|z_{n+1} - z\| &\leq (1-\alpha_n)\|z_n - z\| + \alpha_n\|h(u_n) - h(u) - \rho(Tu_n - Tu)\| \\ &\leq (1-\alpha_n)\|z_n - z\| + \alpha_n\left\{k + \sqrt{1 - 2\rho\alpha + \beta^2\rho^2}\right\}\|u_n - u\|. \end{aligned} \tag{27}$$

Also from (24), (20) and the Lipschitz continuity of the projection operator P_{K_r} with constant δ, we have

$$\|u_n - u\| = \|P_{K_r}z_n - P_{K_r}z\| \leq \delta\|z_n - z\|. \tag{28}$$

Combining (27), and (28), we have

$$\|z_{n+1} - z\| \leq (1-\alpha_n)\|z_n - z\| + \alpha_n\theta\|z_n - z\|. \tag{29}$$

From (12), we see that $\theta < 1$ and consequently

$$\begin{aligned}\|z_{n+1} - z\| &\leq (1-\alpha_n)\|z_n - z\| + \alpha_n\theta\|z_n - z\| \\ &= [1-(1-\theta)\alpha_n]\|z_n - z\| \\ &\leq \prod_{i=0}^{n}[1-(1-\theta)\alpha_i]\|z_0 - z\|.\end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta > 0$, we have $\lim_{n\to\infty}\prod_{i=0}^{n}[1-(1-\theta)\alpha_i] = 0$. Consequently the sequence $\{z_n\}$ convergences strongly to z in H, the required result. □

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