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DEGREE 4 COVERINGS OF ELLIPTIC CURVES BY GENUS 2 CURVES

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Abstract. Genus two curves covering elliptic curves have been the object of study of many articles. For a fixed degree n the subloci of the moduli space $\mathcal{M}_2$ of curves having a degree n elliptic subcover has been computed for $n = 3, 5$ and discussed in detail for n odd; see [17, 22, 3, 4]. When the degree of the cover is even the case in general has been treated in [16]. In this paper we compute the sublocus of $\mathcal{M}_2$ of curves having a degree 4 elliptic subcover.

1. Introduction

Let $\psi : C \to E$ be a degree n covering of an elliptic curve E by a genus two curve C. Let $\pi_C : C \longrightarrow \mathbb{P}^1$ and $\pi_E : E \longrightarrow \mathbb{P}^1$ be the natural degree 2 projections. There is $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ such that the diagram commutes.

$$
\begin{array}{ccc}
C & \xrightarrow{\pi_C} & \mathbb{P}^1 \\
\psi \downarrow & & \downarrow \phi \\
E & \xrightarrow{\pi_E} & \mathbb{P}^1
\end{array}
\tag{1}
$$

The ramification of induced coverings $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ can be determined in detail; see [16] for details. Let σ denote the fixed ramification of $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$. The Hurwitz space of such covers is denoted by $\mathcal{H}(\sigma)$. For each covering $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ (up to equivalence) there is a unique genus two curve C (up to isomorphism). Hence, we have a map

$$\begin{aligned} \Phi: \ \mathcal{H}(\sigma) &\to \mathcal{M}_2 \\ [\phi] &\to [C]. \end{aligned} \tag{2}$$

We denote by $\mathcal{L}_n(\sigma)$ the image of $\mathcal{H}(\sigma)$ under this map. The main goal of this paper is to study $\mathcal{L}_4(\sigma)$.

2. Preliminaries

Most of the material of this section can be found in [23]. Let C and E be curves of genus 2 and 1, respectively. Both are smooth, projective curves defined over k, $char(k) = 0$. Let $\psi : C \longrightarrow E$ be a covering of degree n. From the Riemann-Hurwitz formula, $\sum_{P\in C} (e_\psi(P) - 1) = 2$ where $e_\psi(P)$ is the ramification index of points $P \in C$, under ψ. Thus, we have two points of ramification index 2 or one point of ramification index 3. The two points of ramification index 2 can be in the same fiber or in different fibers. Therefore, we have the following cases of the covering ψ:

Case I: There are P_1, $P_2 \in C$, such that $e_\psi(P_1) = e_\psi(P_2) = 2$, $\psi(P_1) \neq \psi(P_2)$, and $\forall P \in C \setminus \{P_1, P_2\}$, $e_\psi(P) = 1$.

Case II: There are P_1, $P_2 \in C$, such that $e_\psi(P_1) = e_\psi(P_2) = 2$, $\psi(P_1) = \psi(P_2)$, and $\forall P \in C \setminus \{P_1, P_2\}$, $e_\psi(P) = 1$.

Case III: There is $P_1 \in C$ such that $e_\psi(P_1) = 3$, and $\forall P \in C \setminus \{P_1\}$, $e_\psi(P) = 1$.

In case I (resp. II, III) the cover ψ has 2 (resp. 1) branch points in E.

Denote the hyperelliptic involution of C by w. We choose $\mathcal{O}$ in E such that w restricted to$E$ is the hyperelliptic involution on E. We denote the restriction of w on E by v, $v(P) = -P$. Thus, $\psi \circ w = v \circ \psi$. E[2] denotes the group of 2-torsion points of the elliptic curve E, which are the points fixed by v. The proof of the following two lemmas is straightforward and will be omitted.

Lemma 1. *a) If $Q \in E$, then $\forall P \in \psi^{-1}(Q)$, $w(P) \in \psi^{-1}(-Q)$.*
b) For all $P \in C$, $e_\psi(P) = e_\psi(w(P))$.

Let W be the set of points in C fixed by w. Every curve of genus 2 is given, up to isomorphism, by a binary sextic, so there are 6 points fixed by the hyperelliptic involution w, namely the Weierstrass points of C. The following lemma determines the distribution of the Weierstrass points in fibers of 2-torsion points.

Lemma 2. *The following hold:*

1. $\psi(W) \subset E[2]$
2. *If n is an even number then for all $Q \in E[2]$, $\#(\psi^{-1}(Q) \cap W) = 0 \mod (2)$*

Let $\pi_C : C \longrightarrow \mathbb{P}^1$ and $\pi_E : E \longrightarrow \mathbb{P}^1$ be the natural degree 2 projections. The hyperelliptic involution permutes the points in the fibers of π_C and π_E. The ramified points of π_C, π_E are respectively points in W and $E[2]$ and their ramification index is 2. There is $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ such that the diagram commutes.

$$\begin{array}{ccc} C & \xrightarrow{\pi_C} & \mathbb{P}^1 \\ \psi \downarrow & & \downarrow \phi \\ E & \xrightarrow{\pi_E} & \mathbb{P}^1 \end{array} \tag{3}$$

Next, we will determine the ramification of induced coverings $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$. First we fix some notation. For a given branch point we will denote the ramification of points in its fiber as follows. Any point P of ramification index m is denoted by (m). If there are k such points then we write $(m)^k$. We omit writing symbols for unramified points, in other words $(1)^k$ will not be written. Ramification data between two branch points will be separated by commas. We denote by $\pi_E(E[2]) = \{q_1, \ldots, q_4\}$ and $\pi_C(W) = \{w_1, \ldots, w_6\}$.

Let us assume now that $deg(\psi) = n$ is an even number. Then the generic case for $\psi : C \longrightarrow E$ induce the following three cases for $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$:

I: $\left((2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n}{2}}, (2)\right)$

II: $\left((2)^{\frac{n-4}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2)\right)$

III: $\left((2)^{\frac{n-6}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2)\right)$

Each of the above cases has the following degenerations (two of the branch points collapse to one)

I: (1) $\left((2)^{\frac{n}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n}{2}}\right)$

(2) $\left((2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (4)(2)^{\frac{n-6}{2}}, (2)^{\frac{n}{2}}\right)$

(3) $\left((2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (4)(2)^{\frac{n-4}{2}}\right)$

(4) $\left((3)(2)^{\frac{n-4}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n}{2}}\right)$

II: (1) $\left((2)^{\frac{n-2}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}\right)$

(2) $\left((2)^{\frac{n-4}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}\right)$

(3) $\left((4)(2)^{\frac{n-8}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}\right)$

(4) $\left((2)^{\frac{n-4}{2}}, (4)(2)^{\frac{n-6}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}\right)$

(5) $\left((2)^{\frac{n-4}{2}}, (2)^{\frac{n-2}{2}}, (2)^{\frac{n-4}{2}}, (2)^{\frac{n}{2}}\right)$

(6) $\left((3)(2)^{\frac{n-6}{2}}, (2)^{\frac{n-2}{2}}, (4)(2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}\right)$

(7) $\left((2)^{\frac{n-4}{2}}, (3)(2)^{\frac{n-4}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}\right)$

III: (1) $\left((2)^{\frac{n-4}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (4)(2)^{\frac{n}{2}}\right)$

(2) $\left((2)^{\frac{n-6}{2}}, (4)(2)^{\frac{n-4}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}\right)$

(3) $\left((2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (4)(2)^{\frac{n-10}{2}}\right)$

(4) $\left((3)(2)^{\frac{n-8}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}, (2)^{\frac{n}{2}}\right)$

For details see [16].

3. Degree 4 case

In this section we focus on the case $\deg(\phi) = 4$. The goal is to determine all ramifications σ and explicitly compute $\mathcal{L}_4(\sigma)$.

There is one generic case and one degenerate case in which the ramification of $\deg(\phi) = 4$ applies, as given by the above possible ramification structures:

i) $(2, 2, 2, 2^2, 2)$ (generic)
ii) $(2, 2, 2, 4)$ (degenerate)

4. Computing the locus $\mathcal{L}_4$ in $\mathcal{M}_2$

4.1. **Non-degenerate case.** Let $\psi : C \longrightarrow E$ be a covering of degree 4, where C is a genus 2 curve and E is an elliptic curve. Let ϕ be the Frey-Kani covering with $deg(\phi) = 4$ such that $\phi(1) = 0$, $\phi(\infty) = \infty$, $\phi(p) = \infty$ and the roots of $f(x) = x^2 + ax + b$ be in the fiber of 0. In the following figure, bullets (resp., circles) represent places of ramification index 2 (resp., 1).

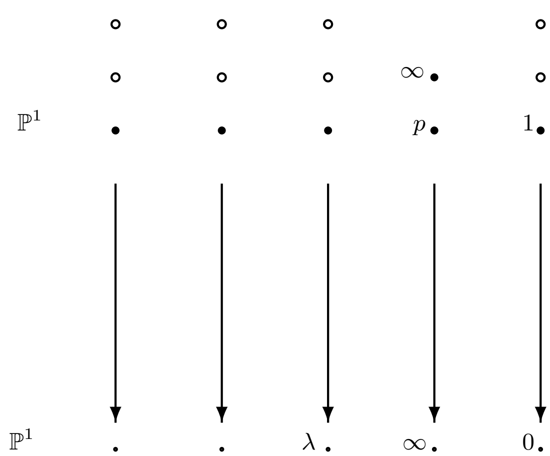

Figure 1. Degree 4 covering for generic case

Then the cover can be given by

$$\phi(x) = \frac{k(x-1)^2(x^2+b)}{(x-p)^2}.$$

Let λ be a 2-torsion point of E. To find λ, we solve

$$\phi(x) - \lambda = 0. \tag{4}$$

According to this ramification we should have 3 solutions for λ, say $\lambda_1, \lambda_2, \lambda_3$. The discriminant of the Eq. (4) gives branch points for the points with ramification index 2. So we have the following relation for λ, with $p \neq 1$.

$$\begin{aligned} &\left(-b-p^2\right)\lambda^3 + \left(2\,kp^2 - 18\,kbp + 16\,kp^4 - 16\,kp^3 + 3\,kb^2 + 3\,kb + 20\,kbp^2\right)\lambda^2 \\ &+ (-3\,k^2b + 21\,k^2b^2 - 36\,k^2b^2p - 3\,k^2b^3 - 20\,k^2bp^2 + 8\,k^2b^2p^2 + 18\,k^2bp \\ &- k^2p^2)\lambda + k^3b + k^3b^4 + 3\,k^3b^2 + 3\,k^3b^3 = 0. \end{aligned} \tag{5}$$

Using Eq.(4) and Eq.(5) we find the degree 12 equation with 2 factors. One of them with degree 6 corresponds to the equation of genus 2 curve and the other corresponds to the double roots in the fiber of λ_1, λ_2 and λ_3.

The equation of genus 2 curve can be written as follows:

$$C : y^2 = a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

where

$$\begin{aligned}
a_6 =& p^2 + b \\
a_5 =& 4\,p^3 - 6\,p^2 + 4\,pb - 6\,b \\
a_4 =& -4\,p^4 - 10\,p^3 + (-5\,b + 13)\,p^2 - 8\,pb + 12\,b \\
a_3 =& 12\,p^4 + (4 + 6\,b)\,p^3 + (-12 + 12\,b)\,p^2 + \left(8\,b^2 - 6\,b\right)p - 8\,b - 8\,b^2 \\
a_2 =& (-11 - 4\,b)\,p^4 + (-20\,b + 6)\,p^3 + \left(4 + 13\,b - 12\,b^2\right)p^2 + 10\,pb + 12\,b^2 \\
a_1 =& (14\,b + 2)\,p^4 + \left(6\,b^2 - 4 + 4\,b\right)p^3 + \left(-24\,b + 6\,b^2\right)p^2 + \left(-6\,b^2 + 4\,b\right)p - 6\,b^2 \\
a_0 =& \left(-b^2 + 1 - 11\,b\right)p^4 + \left(14\,b - 2\,b^2\right)p^3 - 2\,bp^2 + 2\,b^2p + b^2.
\end{aligned}$$

Notice that we write the equation of genus 2 curve in terms of only 2 unknowns. We denote the Igusa invariants of C by J_2, J_4, J_6, and J_{10}. The absolute invariants of C are given in terms of these classical invariants:

$$i_1 = 144\frac{J_4}{J_2^2}, \quad i_2 = -1728\frac{J_2J_4 - 3J_6}{J_2^3}, \quad i_3 = 486\frac{J_{10}}{J_2^5}.$$

Two genus 2 curves with $J_2 \neq 0$ are isomorphic if and only if they have the same absolute invariants. Notice that these invariants of our genus 2 curve are polynomials in p and b. By using a computational symbolic package (as Maple) we eliminate p and b to determine the equation for the non-degenerate locus $\mathcal{L}_4$. The result is very long. We don't display it here.

5. Degenerate Case

Notice that only one degenerate case can occur when $n = 4 : (2, 2, 2, 4)$. In this case one of the Weierstrass points has ramification index 3, so the cover is totally ramified at this point.

Let the branch points be 0, 1, λ, and ∞, where ∞ corresponds to the element of index 4. Then, above the fibers of 0, 1, λ lie two Weierstrass points. The two Weierstrass points above 0 can be written as the roots of a quadratic polynomial $x^2 + ax + b$; above 1, they are the roots of $x^2 + px + q$; and above λ, they are the roots of $x^2 + sx + t$. This gives us an equation for the genus 2 curve C:

$$C : y^2 = (x^2 + ax + b)(x^2 + px + q)(x^2 + sx + t).$$

The four branch points of the cover ϕ are the 2-torsion points $E[2]$ of the elliptic curve E, allowing us to write the elliptic subcover as

$$E : y^2 = x(x-1)(x-\lambda).$$

The cover $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is Frey-Kani covering and is given by

$$\phi(x) = cx^2(x^2 + ax + b).$$

Using $\phi(1) = 1$, we get $c = \frac{1}{1+a+b}$. Then,

$$\phi(x) - 1 = c(x-1)^2(x^2 + px + q).$$

This implies that $\phi'(1) = 0$, so we get $c(4 + 3a + 2b) = 0$. Since c cannot be 0, we must have $4 + 3a + 2b = 0$, which implies $a = \frac{-2(b+2)}{3}$. Combining this with our equation for c, we get $c = \frac{3}{b-1}$.

Now, since $\phi(x) - 1 - c(x-1)^2(x^2 + px + q) = 0$, we want all of the coefficients of this polynomial to be identically 0; thus

$$p = \frac{2(1-b)}{3}, q = \frac{1-b}{3}.$$

Finally, we consider the fiber above λ. We write

$$\phi(x) - \lambda = c(x-r)^2(x^2 + sx + t).$$

Similar to above, we set the coefficients of the polynomial to 0 to get:

$$\lambda = \frac{b^3(4-b)}{16(b-1)}, \quad r = \frac{b}{2}, \quad s = \frac{b-4}{3}, \quad t = \frac{b(b-4)}{12}.$$

Hence we have C and E with equations:

$$\begin{aligned} C: \quad y^2 &= \left(\frac{1-b}{3} + \frac{2}{3}(1-b)x + x^2\right)\left(\frac{1}{12}(b-4)b + \frac{1}{3}(b-4)x + x^2\right) \\ &\quad \left(b - \frac{2}{3}(b+2)x + x^2\right) \\ E: \quad v^2 &= u(u-1)\left(u - \frac{b^3(4-b)}{16(b-1)}\right) \end{aligned} \tag{6}$$

where the corresponding discriminants of the right sides must be non-zero. Hence,

$$\Delta_C := b(b-4)(b-2)(b-1)(2+b) \neq 0 \tag{7}$$

$$\Delta_E := \frac{(b-4)^2(b-2)^6 b^6 (b+2)^2}{65536(b-1)^4} \neq 0. \tag{8}$$

From here on, we consider the additional restriction on b that it does not solve $J_2 = 0$, that is,

$$J_2 = -\frac{5}{486}(256 - 384b - 4908b^2 + 5068b^3 - 1227b^4 - 24b^5 + 4b^6) \neq 0. \tag{9}$$

The case when $J_2 = 0$ is considered separately. We can eliminate b from this system of equations by taking the numerators of $i_j - i_j(b)$ and setting them equal to 0, where i_j are absolute invariants of genus 2 curve.

Thus, we have 3 polynomials in b, i_1, i_2, i_3. We eliminate b using the method of resultants and get the following:

$$\begin{aligned} &3652054494822999 - 312800728170302145 i_1 - 247728254774362875 i_1^2 \\ &+3039113062253125 i_1^3 - 522534367747902600 i_2 - 28017734537115000 i_1 i_2 \\ &-238234372300000 i_2^2 = 0 \end{aligned} \tag{10}$$

and the other equation

$$\begin{aligned} &1158391804615233525 i_1 - 17653298856896250 i_1^2 + 100894442906250 i_1^3 \\ &-256292578125 i_1^4 + 244140625 i_1^5 - 32389016798910273266880000 i_3 \\ &-1487967222528890496000000 i_1 i_3 - 40609431102258000000000 i_1^2 i_3 \\ &-16677181699666569 + 347405361918358396861440000000000 i_3^2 = 0 \end{aligned} \tag{11}$$

These equations determine the degenerate locus $\mathcal{L}_4'$ when $J_2 \neq 0$.

When $J_2 = 0$, we must resort to the a-invariants of the genus 2 curve. These invariants are defined as

$$a_1 = \frac{J_4 J_6}{J_{10}}, \qquad a_2 = \frac{J_{10} J_6}{J_4^4}.$$

Two genus 2 curves with $J_2 = 0$ are isomorphic iff their a-invariants are equal. For our genus 2 curve,

$$J_4 = \frac{1}{5184}\big(65536 - 196608b - 307200b^2 + 1218560b^3 - 834288b^4 - 294432b^5 \\ +456600b^6 - 73608b^7 - 52143b^8 + 19040b^9 - 1200b^{10} - 192b^{11} + 16b^{12}\big)$$

It can be guarantee that J_4 and J_2 are not simultaneously 0 because the resultant of these two polynomials in b is

$$\frac{1178497805152239570764667289600000000000 0}{4239115827521620351429443320 1},$$

so there are no more subcases. We want to eliminate b from the set of equations:

$$\begin{aligned} J_2 &= 0 \\ a_1 - a_1(b) &= 0 \\ a_2 - a_2(b) &= 0. \end{aligned}$$

Similar to what we did above with the i-invariants, we take resultants of combinations of these and set them equal to 0. Doing so tells us

$$\begin{aligned} &20a_1 - 55476394831 = 0 \\ &1022825924657928a_2 - 522665 = 0. \end{aligned}$$

So in other words, if C is a genus 2 curve with a degree 4 elliptic subcover with $J_2 = 0$, then

$$a_1 = \frac{55476394831}{20}, \quad a_2 = \frac{522665}{1022825924657928}.$$

So up to isomorphism, this is the only genus 2 curve with degree 4 elliptic subcover with $J_2 = 0$. In this case the equation of the genus 2 curve is given by Eq.(6), where b is given by the following:

$$b = \frac{2\alpha + \sqrt{429\alpha^2 + 60123\alpha + \beta}}{2\alpha} \tag{12}$$

with $\alpha = \sqrt[3]{2837051 + 9408\, i\sqrt{5}}$ and $\beta = 8511153 + 28224\, i\sqrt{5}$. We summarize the above results in the following theorem.

Theorem 1. *Let C be a genus 2 curve with a degree 4 degenerate elliptic subcover. Then C is isomorphic to the curve given by Eq.(6) where b satisfies Eq.(12) or its absolute invariants satisfy Eq. (10) and Eq. (11).*

Remark 1. *The genus 2 curve, when $J_2 = 0$, is not defined over the rational.*

Remark 2. *When the genus 2 curve has non zero J_2 invariant the j invariant of the elliptic curve satisfies the following equation:*

$$
\begin{aligned}
0 =&(262144000000000\, J_4{}^4 - 14332985344000000\, J_2{}^2 J_4{}^3 - 15871355368243200\, J_2{}^6 J_4 \\
&+ 1586874322944\, J_2{}^8 + 26122821304320000\, J_2{}^4 J_4{}^2) j^2 + (-2535107603331605760\, J_2{}^8 \\
&+ 25102192337335536076800\, J_2{}^6 J_4 - 16478102426419200000000\, J_4{}^4 \\
&+ 90675809529498685440000\, J_2{}^4 J_4{}^2 - 36316352208339763200000\, J_2{}^2 J_4{}^3) j \\
&+ 258949145865976645040640000000\, J_4{}^4 - 203482361042468209670400000000\, J_2{}^2 J_4{}^3 \\
&+ 3986271076680255204562\, 5 J_2{}^8 - 194338063261907411418000000\, J_2{}^6 J_4 \\
&+ 325954300436274690741600000\, J_2{}^4 J_4{}^2.
\end{aligned}
$$

5.1. **Genus 2 curves with degree 4 elliptic subcovers and extra automorphisms in the degenerate locus of $\mathcal{L}_4$.** In any characteristic different from 2, the automorphism group Aut(C) is isomorphic to one of the groups : C_2, C_{10}, V_4, D_8, D_{12}, $C_3 \rtimes D_8$, $GF_2(3)$, or 2^+S_5; See [21] for the description of each group. We have the following lemma.

Lemma 3. (a) *The locus $\mathcal{L}_2$ of genus 2 curves C which have a degree 2 elliptic subcover is a closed subvariety of $\mathcal{M}_2$. The equation of $\mathcal{L}_2$ is given by*

(13)
$$
\begin{aligned}
0 =\ & 8748 J_{10} J_2^4 J_6^2 - 507384000 J_{10}^2 J_4^2 J_2 - 19245600 J_{10}^2 J_4 J_2^3 - 592272 J_{10} J_4^4 J_2^2 \\
&+ 77436 J_{10} J_4^3 J_2^4 - 3499200 J_{10} J_2 J_6^3 + 4743360 J_{10} J_4^3 J_2 J_6 - 870912 J_{10} J_4^2 J_2^3 J_6 \\
&+ 3090960 J_{10} J_4 J_2^2 J_6^2 - 78 J_2^5 J_4^5 - 125971200000 J_{10}^3 - 81 J_2^3 J_6^4 + 1332 J_2^4 J_4^4 J_6 \\
&+ 384 J_4^6 J_6 + 41472 J_{10} J_4^5 + 159 J_4^6 J_2^3 - 236196 J_{10}^2 J_2^5 - 80 J_4^7 J_2 - 47952 J_2 J_4 J_6^4 \\
&+ 104976000 J_{10}^2 J_2^2 J_6 - 1728 J_4^5 J_2^2 J_6 + 6048 J_4^4 J_2 J_6^2 - 9331200 J_{10} J_4^2 J_6^2 - J_2^7 J_4^4 \\
&+ 12 J_2^6 J_4^3 J_6 + 29376 J_2^2 J_4^2 J_6^3 - 8910 J_2^3 J_4^3 J_6^2 - 2099520000 J_{10}^2 J_4 J_6 + 31104 J_6^5 \\
&- 6912 J_4^3 J_6^3 4 - 5832 J_{10} J_2^5 J_4 J_6 - 54 J_2^5 J_4^2 J_6^2 + 108 J_2^4 J_4 J_6^3 + 972 J_{10} J_2^6 J_4^2.
\end{aligned}
$$

(b) *The locus $\mathcal{M}_2(D_8)$ of genus 2 curves C with Aut(C) $\equiv D_8$ is given by the equation of $\mathcal{L}_2$ and*

$$0 = 1706 J_4^2 J_2^2 + 2560 J_4^3 + 27 J_4 J_2^4 - 81 J_2^3 J_6 - 14880 J_2 J_4 J_6 + 28800 J_6^2. \tag{14}$$

(c) *The locus $\mathcal{M}_2(D_{12})$ of genus 2 curves C with Aut(C) $\equiv D_{12}$ is*

$$0 = -J_4 J_2^4 + 12 J_2^3 J_6 - 52 J_4^2 J_2^2 + 80 J_4^3 + 960 J_2 J_4 J_6 - 3600 J_6^2 \tag{15}$$

$$
\begin{aligned}
0 =& -864 J_{10} J_2^5 + 3456000 J_{10} J_4^2 J_2 - 43200 J_{10} J_4 J_2^3 - 2332800000 J_{10}^2 \\
&- J_4^2 J_2^6 - 768 J_4^4 J_2^2 + 48 J_4^3 J_2^4 + 4096 J_4^5.
\end{aligned} \tag{16}
$$

We will refer to the locus of genus 2 curves C with Aut(C) $\equiv D_{12}$ (resp., Aut(C) $\equiv D_8$) as the D_{12}-locus (resp., D_8-locus).

Equations (10), (11), and (13) determine a system of 3 equations in the 3 i-invariants. The set of possible solutions to this system contains 20 rational points and 8 irrational or complex points (there may be more possible solutions, but finding them involves the difficult task of solving a degree 15 or higher polynomial).

Among the 20 rational solutions, there are four rational points which actually solve the system.

$$(i_1, i_2, i_3) = \left(\frac{102789}{12005}, \frac{-73594737}{2941225}, \frac{531441}{28247524900000} \right)$$
$$(i_1, i_2, i_3) = \left(\frac{66357}{9245}, \frac{-892323}{46225}, \frac{7776}{459401384375} \right)$$
$$(i_1, i_2, i_3) = \left(\frac{235629}{1156805}, \frac{-28488591}{214008925}, \frac{53747712}{80459143207503125} \right)$$
$$(i_1, i_2, i_3) = \left(\frac{1078818669}{383775605}, \frac{-77466710644803}{16811290377025}, \frac{1356226634181762}{161294078381836186878125} \right).$$

Of these four points, only the first one lies on the D_{12}-locus, and none lie on the D_8-locus, so the other three curves have automorphism groups isomorphic to V_4 (See Remark 3 for their equations). We have the following proposition.

Proposition 1. *There is exactly one genus 2 curve C defined over $\mathbb{Q}$ (up to $\mathcal{C}$-isomorphism) with a degree 4 elliptic subcover which has an automorphism group D_{12} namely the curve*

$$C = 100X^6 + 100X^3 + 27$$

and no such curves with automorphism group D_8.

Proof. From above discussion there is exactly one rational point which lies on the D_{12}-locus and three rational points which lies on the V_4-locus. Furthermore we have the fact that $\mathrm{Aut}(C) \equiv D_{12}$ if and only if C is isomorphic to the curve given by $Y^2 = X^6 + X^3 + t$ for some $t \in k$; see [19] for more details.

Suppose the equation of the D_{12} case is $Y^2 = X^6 + X^3 + t$. We want to find t. We can calculate the i-invariants in terms of t accordingly, so we get a system of equations, $i_j - i_j(t) = 0$ for $j \in \{1, 2, 3\}$. Those equations simplify to the following:

$$\begin{aligned}
0 &= 1600 i_1 t^2 - 80 i_1 t + i_1 - 6480 t^2 - 1296 t \\
0 &= 64000 i_2 t^3 - 4800 i_2 t^2 + 120 i_2 t - i_2 + 233280 t^3 + 303264 t^2 - 11664 t \\
0 &= 1638400000 i_3 t^5 - 204800000 i_3 t^4 + 10240000 i_3 t^3 - 256000 i_3 t^2 \\
&\quad + 3200 i_3 t - 16 i_3 + 729 t^2 + 34992 t^2 - 46656 t^5 - 8748 t^3.
\end{aligned}$$

Replacing our i-invariants into the above system of equations we get:

$$\begin{aligned}
0 &= 86670000\, t^2 - 23781600\, t + 102789 \\
0 &= -4023934200000\, t^3 + 1245222396000\, t^2 - 43137816840\, t + 73594737 \\
0 &= -82315363050000000\, t^5 + 61770534511500000\, t^4 - 15443994116835000\, t^3 \\
&\quad + 1287019350200250\, t^2 + 106288200\, t - 531441.
\end{aligned}$$

There is only root those three polynomials share: $t = \frac{27}{100}$. Thus, there is exactly one genus 2 curve C defined over Q (up to Q-isomorphism) with a degree 4 elliptic subcover which has an automorphism group D_{12}

$$C: \quad y^2 = 100X^6 + 100X^3 + 27$$

Similarly, we show that there are no such curves with automorphism group D_8. $\square$

Remark 3. *There are at least three genus 2 curves defined over* $\mathbb{Q}$ *with automorphism group* V_4. *The equations of these curves are given by the followings:*

Case 1: $(i_1, i_2, i_3) = \left(\frac{66357}{9245}, \frac{-892323}{46225}, \frac{7776}{459401384375}\right)$

$$
\begin{aligned}
C : y^2 = {} & 1432139730944\, x^6 + 34271993769359360\, x^5 + 267643983706245216000\, x^4 \\
& + 1267919172426862313120000\, x^3 + 2394555897022488621383535 0000\, x^2 \\
& + 2743306661626491537935993804750 00\, x + 1025623291911204380755800513010015625.
\end{aligned}
$$

Case 2: $(i_1, i_2, i_3) = \left(\frac{235629}{1156805}, \frac{-28488591}{214008925}, \frac{53747712}{80459143207503125}\right)$

$$
\begin{aligned}
C : y^2 = {} & 41871441565158964373437321767075023159296\, x^6 \\
& + 156000358914872008908017177004915818496000\, x^5 \\
& + 899442975326825232869917531312226304000000 0\, x^4 \\
& + 178575374038215615794800535745331200000000 00\, x^3 \\
& + 775018151562516781352226536816640000000000 00000\, x^2 \\
& + 115824938236869101167923689937600000000000 00000\, x \\
& + 2678752767946851427317565520095988845 8251953125.
\end{aligned}
$$

Case 3: $(i_1, i_2, i_3) = \left(\frac{1078818669}{383775605}, \frac{-77466710644803}{16811290377025}, \frac{1356226634181762}{161294078381836186878125}\right)$

$$
\begin{aligned}
C : y^2 = {} & 9224408124038149308993379217084884661375653227720704\, x^6 \\
& + 3730758767668984877725129604888152322035364826481920000\, x^5 \\
& + 113852328380343991240386194428199809225534591301754000 0000\, x^4 \\
& + 1894250490477817846232618952385906586748412048834575000 00000\, x^3 \\
& + 762125205676149190950324121543822184439329394838171289062500 00\, x^2 \\
& + 1671729419207307054705692151510108869289820883462418090820312500 0\, x \\
& + 2766888989045448736067444316860942956954296161559210811614990234375.
\end{aligned}
$$

We summarize by the following:

Theorem 2. *Let* $\psi : C \to E$ *be a degree 4 covering of an elliptic curve by a genus 2 curve. Then the following hold:*

i) In the generic case the equation of C *can be written as follows:*

$$C : y^2 = a_6 x^6 + a_5 x^5 + \cdots + a_1 x + a_0$$

where

$$
\begin{aligned}
a_6 &= p^2 + b \\
a_5 &= 4\,p^3 - 6\,p^2 + 4\,pb - 6\,b \\
a_4 &= -4\,p^4 - 10\,p^3 + (-5\,b + 13)\,p^2 - 8\,pb + 12\,b \\
a_3 &= 12\,p^4 + (4 + 6\,b)\,p^3 + (-12 + 12\,b)\,p^2 + \left(8\,b^2 - 6\,b\right)p - 8\,b - 8\,b^2 \\
a_2 &= (-11 - 4\,b)\,p^4 + (-20\,b + 6)\,p^3 + \left(4 + 13\,b - 12\,b^2\right)p^2 + 10\,pb + 12\,b^2 \\
a_1 &= (14\,b + 2)\,p^4 + \left(6\,b^2 - 4 + 4\,b\right)p^3 + \left(-24\,b + 6\,b^2\right)p^2 + \left(-6\,b^2 + 4\,b\right)p - 6\,b^2 \\
a_0 &= \left(-b^2 + 1 - 11\,b\right)p^4 + \left(14\,b - 2\,b^2\right)p^3 - 2\,bp^2 + 2\,b^2 p + b^2.
\end{aligned}
$$

ii) In the degenerate case the equation of $\mathcal{L}_4'$ is given by

$$
\begin{aligned}
&1541086152812576000\, J_2^{\,2} J_4^{\,2} - 228353122323609600000\, J_2\, J_4\, J_6 + 5009676947631\, J_2^{\,6} \\
&-8782271900467200000\, J_6^{\,2} + 1176812184652746480\, J_2^{\,4} J_4 + 1244820710298880000\, J_4^{\,3} \\
&-3715799948429529600\, J_2^{\,3} J_6 = 0
\end{aligned}
$$

$$
\begin{aligned}
&1866265600000000\, J_2^{\,2} J_4^{\,4} + 138962144767343358744576000000000\, J_{10}^{\,2} + 282429536481\, J_2^{\,10} \\
&+6199238007360000\, J_2^{\,6} J_4^{\,2} - 256000000000000\, J_4^{\,5} - 2824915237592400\, J_2^{\,8} J_4 \\
&+2665762699498787923200000\, J_2^{\,5} J_{10} - 5102020224000000\, J_2^{\,4} J_4^{\,3} \\
&+69306762414520320000000000\, J_2\, J_4^{\,2} J_{10} + 1763516708182388736000000000\, J_2^{\,3} J_4\, J_{10} = 0
\end{aligned}
$$

iii) The intersection $\mathcal{L}_4' \cap \mathcal{M}_2(D_8) = \emptyset$ and the intersection $\mathcal{L}_4' \cap \mathcal{M}_2(D_{12})$ contains a single point, namely the curve

$$C: \quad y^2 = 100X^6 + 100X^3 + 27$$

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