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OTHER REPRESENTATIONS OF THE RIEMANN ZETA FUNCTION AND AN ADDITIONAL REFORMULATION OF THE RIEMANN HYPOTHESIS

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Abstract. New expansions for some functions related to the Zeta function in terms of the Pochhammer polynomials are given (coefficients b_k, d_k and $\hat{d}_k$). In some formal limit our expansion b_k obtained via the alternating series gives the regularized expansion of Maslanka for the Zeta function. The real and the imaginary part of the function on the critical line is obtained with a good accuracy up to $\Im(s) = t < 35$.

Then, we give the expansion (coefficient $\hat{d}_k$) for the derivative of $\ln[(s-1)\zeta(s)]$. The critical function of the derivative, whose bounded values for $\Re(s) > \frac{1}{2}$ at large values of k should ensure the truth of the Riemann Hypothesis (RH), is obtained either by means of the primes or by means of the zeros (trivial and non-trivial) of the Zeta function. In a numerical experiment performed up to high values of k i.e. up to $k = 10^{14}$ we obtain a very good agreement between the two functions, with the emergence of fourteen oscillations with stable amplitude.

1. Introduction

Lately there has been new interest in the study of the expansion of the Zeta function via the Pochhammer polynomials. This is related to the original idea of Riesz [17] and of Hardy-Littlewood [13] at the beginning of the last century. In pioneering works, Maslanka obtained a regularized expansion for the Zeta function (with coefficients A_k) [14] and Baez-Duarte an expansion for the reciprocal of the Zeta function (with coefficients c_k) for the Riesz case [2, 4]. Other cases of interest have also recently been studied [1, 8, 9, 10, 15, 18]. As pointed out in [4], the discrete version by means of the Pochhammer polynomials $P_k(s)$, where $s = \sigma + it$ is the complex variable and k is an integer, has advantages especially in the context of numerical experiments in connection with some "kind of verification" that supports the RH may be true.

In this work we first derive a new expansion for the Zeta function in terms of the Pochhammer polynomials via the alternating series (with new coefficients b_k). In some formal limit, a connection with the expansion of Maslanka is also obtained in Section 2. Our expansion is then studied numerically on the critical line where a good agreement with the real function is obtained up to $\Im(s) = t < 35$, with the

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emergence of the first few low zeros. After this value of t, a divergence possibly of numerical nature set on.

In Section 3 we then obtain the expansion for the function $\ln\left[(1-2^{1-s})\zeta(s)\right]$ (with new coefficients d_k) as well as for the derivative of $\ln\left[(s-1)\zeta(s)\right]$ (with new coefficients $\hat{d_k}$) in terms of the two parameters α and β, already introduced in our previous works [5, 6, 7]. The critical function for the derivative (whose boundedness at large k would "ensure" the truth of the RH) is then obtained either with the primes or with the trivial and non-trivial zeros of the Zeta function.

In the numerical experiment for the special case $\alpha=\frac{9}{2}$ and $\beta=4$ up to high values of k, i.e. $k=10^{14}$, the results for the two functions are in very good agreement, both with the emergence of the same fourteen oscillations of stable amplitude of about 0.01 (Section 4).

2. Zeta function representation via the alternating series

In this section we derive a formula for $(1-2^{1-s})\zeta(s)$ similar to the one of Maslanka for $(s-1)\zeta(s)$ [14] and of Baez-Duarte for $[\zeta(s)]^{-1}$ [2, 4].

Here the starting series is convergent for $\Re(s)=\sigma>0$ and the formula is obtained still in terms of the so called Pochhammer polynomials of degree k, in the complex variable $s=\sigma+it$.

$$P_k(s)=\prod_{r=1}^{k}\left(1-\frac{s}{r}\right) \qquad \forall k\in\mathbb{N}^* \quad \text{and} \quad P_0(s)=1 \tag{2.1}$$

We will also use a family of functions with two parameters (α and β) as considered already in our recent works [5, 6, 7]. Since the alternating series is given by:

$$\left(1-2^{1-s}\right)\zeta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} \qquad \forall\Re(s)=\sigma>0 \tag{2.2}$$

we have using the trick as in [2] that:

$$\begin{aligned}\left(1-2^{1-s}\right)\zeta(s) &= \sum_{n=1}^{\infty}\tfrac{(-1)^{n-1}}{n^\alpha}\left(1-\left(1-\tfrac{1}{n^\beta}\right)\right)^{\frac{s-\alpha}{\beta}}\\ &= \sum_{n=1}^{\infty}\tfrac{(-1)^{n-1}}{n^\alpha}\sum_{k=0}^{\infty}(-1)^k\left(1-\tfrac{1}{n^\beta}\right)^k\binom{\frac{s-\alpha}{\beta}}{k}\end{aligned}$$

Since

$$\begin{aligned}(-1)^k\binom{\frac{s-\alpha}{\beta}}{k} &= \tfrac{(-1)^k}{k!}\left(\tfrac{s-\alpha}{\beta}+1-1\right)\cdots\left(\tfrac{s-\alpha}{\beta}+1-k\right)\\ &= \prod_{r=1}^{k}\left(1-\frac{\frac{s-\alpha}{\beta}+1}{r}\right)=P_k\left(\tfrac{s-\alpha}{\beta}+1\right)\end{aligned}$$

we obtain:

$$\begin{aligned}\left(1-2^{1-s}\right)\zeta(s) &= \sum_{k=0}^{\infty}P_k\left(\tfrac{s-\alpha}{\beta}+1\right)\sum_{n=1}^{\infty}\tfrac{(-1)^{n-1}}{n^\alpha}\left(1-\tfrac{1}{n^\beta}\right)^k\\ &= \sum_{k=0}^{\infty}P_k\left(\tfrac{s-\alpha}{\beta}+1\right)\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{n=1}^{\infty}\tfrac{(-1)^{n-1}}{n^{\alpha+\beta j}}\end{aligned} \tag{2.3}$$

Since from (2.2)

$$\left(1-2^{1-(\alpha+\beta j)}\right)\zeta(\alpha+\beta j)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\alpha+\beta j}}$$

substitution in (2.3) gives:
(2.4)

$$\left(1-2^{1-s}\right)\zeta(s)=\sum_{k=0}^{\infty}P_k\Big(\frac{s-\alpha}{\beta}+1\Big)\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(1-2^{1-(\alpha+\beta j)}\right)\zeta(\alpha+\beta j)$$

With the definition

$$b_k:=\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(1-2^{1-(\alpha+\beta j)}\right)\zeta(\alpha+\beta j) \tag{2.5}$$

(2.4) becomes:

$$\left(1-2^{1-s}\right)\zeta(s)=\sum_{k=0}^{\infty}b_kP_k\Big(\frac{s-\alpha}{\beta}+1\Big) \tag{2.6}$$

where $P_0(\frac{s-\alpha}{\beta}+1)=1$ and $b_0=(1-2^{1-\alpha})\zeta(\alpha)$.

The series above, is expected to represent $(1-2^{1-s})\zeta(s)$ for s in some compact subset of the plane as for the Maslanka case [14]. In that case, the central point has been investigated and elucidated by Baez-Duarte [3]. Here many choices of α and β are possible. For $\alpha=\beta=2$ we have the Riesz case [17] and it is the analogon to the regularized version of Maslanka but the representation of the Zeta function is not the same. For $\alpha=1+\delta$ $(\delta\downarrow 0)$ and $\beta=2$ we obtain the Hardy-Littlewood case [13] which was also discussed numerically in a different way using other polynomials [12].

In fact, from Lemma 2.3 of Baez-Duarte [4] which states that at large k:

$$|P_k(s)|\le Ck^{-\Re(s)} \tag{2.7}$$

where C is a constant depending on |s|, we obtain here that:

$$\left|P_k\Big(\frac{s-\alpha}{\beta}+1\Big)\right|\le Ck^{-\left(\frac{\Re(s)-\alpha}{\beta}+1\right)}$$

We thus suspect and expect that the above series represents $(1-2^{1-s})\zeta(s)$ for all $\Re(s)>\frac{1}{2}+\delta,\delta>0$ if we assume $|b_k|\le Dk^{-\gamma}$ with $\gamma\ge\frac{\alpha-1/2-\delta}{\beta}$ at large values of k and for some constant D. In fact with this assumption we have that:

$$\begin{aligned}\left|\left(1-2^{1-s}\right)\zeta(s)\right| &\le \sum_{k=0}^{\infty}\left|b_kP_k(\tfrac{s-\alpha}{\beta}+1)\right|\le \text{const.}\sum_{k=0}^{\infty}k^{-\frac{\alpha-1/2-\delta}{\beta}}k^{-\left(\frac{\Re(s)-\alpha}{\beta}+1\right)}\\ &\le \text{const.}\sum_{k=0}^{\infty}k^{-\left(1+\frac{\Re(s)-1/2-\delta}{\beta}\right)}<\infty\end{aligned}$$

if $\Re(s)>\frac{1}{2}+\delta$.

For $\alpha=\beta=2$ (case of Riesz) we should have $|b_k|\le Dk^{-\frac{3}{4}+\epsilon}$. For the case $\alpha=1$ and $\beta=2$ (case of Hardy-Littlewood) we should have $|b_k|\le Dk^{-\frac{1}{4}+\epsilon}$. Another case of interest is the one where $\alpha=\frac{3}{2}$ and $\beta=1$. In this case one should have $|b_k|\le Dk^{-1+\epsilon}$.

Of interest also, is the limiting case of large values of β, where barely b_k should behave as $|b_k|\le D$.

For a strong argument (a Theorem) in favour of the validity of the Maslanka representation of $(s-1)\zeta(s)$ in some regions of the complex plane (compact subsets), the reader should consult the work of Baez-Duarte [3] already mentioned and it is expected that using the same methods, the proof of (2.6) may be obtained for all $\Re(s) > \frac{1}{2}$. Here, for our series we limit ourselves to a numerical analysis just illustrating the kind of accuracy of some representations.

Remark 2.1. Let us consider the Riesz case $\alpha = \beta = 2$. We can write:

$$\left(1 - e^{(1-s)\ln 2}\right)\zeta(s) = \sum_{k=0}^{\infty} P_k\left(\frac{s}{2}\right)\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(1 - e^{-(1+2j)\ln 2}\right)\zeta(2+2j)$$

and using Taylor's expansion of e^x, we obtain:

$$(s-1)\,\zeta(s) = \sum_{k=0}^{\infty} A_k P_k\left(\frac{s}{2}\right) \tag{2.8}$$

where

$$A_k = \sum_{j=0}^{k}(-1)^j\binom{k}{j}(2j+1)\,\zeta(2j+2) \tag{2.9}$$

i.e. the representation obtained originally by a different method by Maslanka in a pioneering work [14]. We remark that (2.8) and (2.9) should not be considered as an approximation of our formulas (2.5) and (2.6) and vice versa. (2.5), (2.6) and (2.8), (2.9) are simply two different representations of functions related to the Riemann Zeta function, the first one given by $(s-1)\zeta(s)$, the second one by $(1-2^{1-s})\zeta(s)$.

As an example, for $s = \sigma$ with σ in $[0,1]$, both representations give a good description of the real function $\zeta(\sigma)$ as may easily be computationally checked. We omit here the details.

We now proceed to obtain a representation of $\zeta(s)$ possibly correct on the critical line $s = \frac{1}{2} + it$, with the help of (2.5) and (2.6), in which we are free to set $\alpha = \frac{1}{2}$ and $\beta = i$. Then:

$$\left(1 - 2^{\frac{1}{2}-it}\right)\zeta\left(\frac{1}{2} + it\right) = \sum_{k=0}^{\infty} b_k P_k(t+1) \tag{2.10}$$

where now

$$b_k = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(1 - 2^{\frac{1}{2}-ij}\right)\zeta\left(\frac{1}{2} + ij\right) \tag{2.11}$$

We now check the series in (2.10) restricting k up to 20 for $t \leq 18$ and up to 50 for $t > 18$. We compare the result with the exact functions $\Re\left((1-2^{\overline{s}})\zeta(s)\right)$ and $\Im\left((1-2^{\overline{s}})\zeta(s)\right)$, for $s = \frac{1}{2} + it$ with t up to 40. The plots are given below. The numerical results are satisfactory until $t \cong 35$. We obtain a good qualitative approximation with the emergence of the first five non-trivial zeros (t_i). In Table 1 we obtained the calculated t_i by means of the function "FindRoot" in the software package *Mathematica*.

Remark 2.2. If instead of the value $\beta = i$ we set $\beta = \frac{i}{m}$ (m integer), then it may be verified that (2.6) gives for $t < k$ and $t = \frac{n}{m}$ (n integer) the same values as the true function $\zeta(\frac{1}{2} + it)$. For these cases more analytical as well as numerical studies are needed. Moreover as k is increasing, we note the emergence of strange oscillations propagating from $t = 0$ away. We argue that numerical complexity set on at this point and we have at the moment no answer to this problem. Researchers are invited to give more elucidations and results in this direction.

TABLE 1. The first five non-trivial zeros t_i calculated by means of the real part of $\sum_{k=0}^{20(50)} b_k P_k(t+1)$

	t_i, see Odlyzko [16]	calculated t_i
t_1	14.13472514173469	14.05988000296
t_2	21.02203963877155	21.02212625771
t_3	25.01085758014569	25.01083570045
t_4	30.42487612585951	30.39283277445
t_5	32.93506158773919	32.99863566475

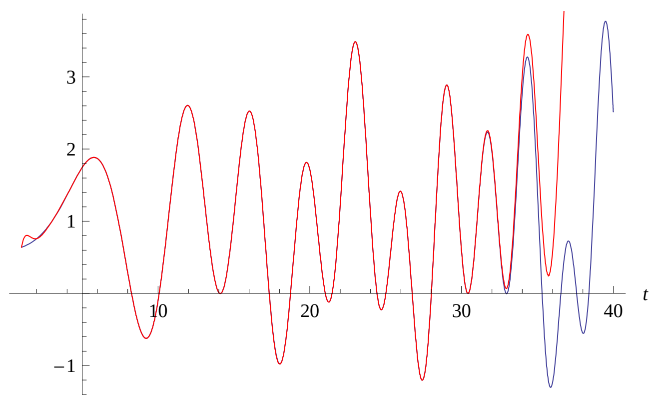

FIGURE 1. The plot of the real part of $\sum_{k=0}^{20(50)} b_k P_k(t+1)$ [red] vs. $\Re\left((1-2^{\overline{s}})\zeta(s)\right)$ [blue]

Remark 2.3. The right hand side of (2.10) is a polynomial in the variable t with complex coefficients. It can be seen as a "characteristic polynomial" associated with some matrix whose coefficients depend on the $b(k)$ i.e. on the values of the Zeta function at integer height j on the critical line. The eigenvalues of the matrix should contain a subset given by the non-trivial zeros of the Zeta function. This may be seen on Figure 1 and on Figure 2 for the first few low zeros where $t \leq 33$.

This concludes the first part of our work. Below, in the second part we develop two new representations of the functions $\ln\left[(1-2^{1-s})\zeta(s)\right]$ and $\frac{d}{ds}\ln\left[(s-1)\zeta(s)\right]$ which may possibly constitute a satisfactory approximation to the exact functions.

3. A REPRESENTATION FOR THE LOGARITHM OF THE ZETA FUNCTION AND AN ADDITIONAL CRITERION FOR THE TRUTH OF THE RH

We will start as before but instead of writing $\zeta(s)$ as a sum, i.e. $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, we will use the Euler product formula to derive a new representation for $\ln\left[(1-2^{1-s})\zeta(s)\right]$, which of course should be carefully investigated by means of

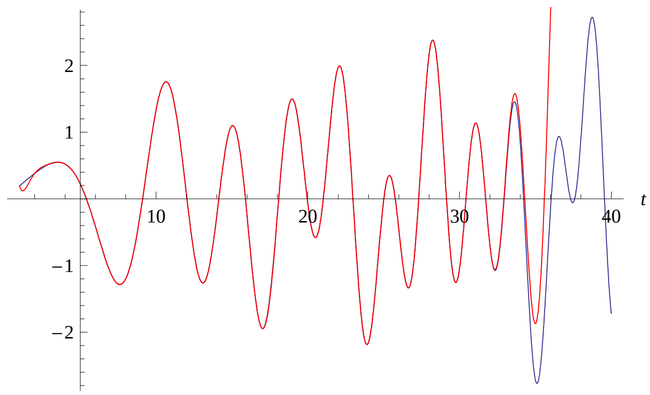

FIGURE 2. The plot of the imaginary part of $\sum_{k=0}^{20(50)} b_k P_k(t+1)$ [red] vs. $\Im\left((1-2^{\overline{s}})\zeta(s)\right)$ [blue]

some numerical experiments. Thus:

$$\ln\left[\left(1-2^{1-s}\right)\zeta(s)\right] = \ln\left[\left(1-2^{1-s}\right)\prod_{p\,\text{prime}}\frac{1}{1-p^{-s}}\right] \qquad \forall\Re(s)>1 \tag{3.1}$$

For any prime p, we have:

$$\ln\left(1-p^{-s}\right) = -\sum_{n=1}^{\infty}\frac{p^{-ns}}{n}$$

so that introducing the parameters α and β as before we have that:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{p^{-\alpha n}}{n}\left(1-\left(1-p^{-\beta n}\right)\right)^{\frac{s-\alpha}{\beta}} &= \sum_{n=1}^{\infty}\frac{p^{-\alpha n}}{n}\sum_{k=0}^{\infty}(-1)^k\left(1-p^{-\beta n}\right)^k\binom{\frac{s-\alpha}{\beta}}{k}\\ &= \sum_{k=0}^{\infty}P_k\left(\tfrac{s-\alpha}{\beta}+1\right)\sum_{n=1}^{\infty}\frac{1}{n}\sum_{j=0}^{k}(-1)^j\binom{k}{j}p^{-(\alpha+\beta j)n}\\ &= \sum_{k=0}^{\infty}P_k\left(\tfrac{s-\alpha}{\beta}+1\right)\sum_{j=0}^{k}(-1)^j\binom{k}{j}\ln\left(1-p^{-(\alpha+\beta j)}\right)\end{aligned}$$

The same treatment for the function $\ln\left(1-2^{1-s}\right)$, gives:

$$\ln\left(1-2^{1-s}\right) = \sum_{k=0}^{\infty}P_k\left(\frac{s-\alpha}{\beta}+1\right)\sum_{j=0}^{k}(-1)^j\binom{k}{j}\ln\left(1-2^{1-(\alpha+\beta j)}\right)$$

where P_k are still the Pochhammer polynomials.

Finally, the representation of $\ln\left[(1-2^{1-s})\zeta(s)\right]$, we propose is given by:

$$\ln\left[\left(1-2^{1-s}\right)\zeta(s)\right] = \sum_{k=0}^{\infty}d_kP_k\left(\frac{s-\alpha}{\beta}+1\right) \tag{3.2}$$

where now:

$$d_k := \sum_{j=0}^{k}(-1)^j\binom{k}{j}\ln\left[\left(1-2^{1-(\alpha+\beta j)}\right)\zeta(\alpha+\beta j)\right] \tag{3.3}$$

Remark 3.1. Another formal derivation of the above equations is the following:

$$\ln\left[\left(1-2^{1-s}\right)\zeta(s)\right]=\ln\left[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}\right]$$

Supposing now that the right hand side may be given as an unknown series $\sum\limits_{r=1}^{\infty}\frac{a_r}{r^s}$ we then have:

$$\begin{aligned}\sum_{r=1}^{\infty}\tfrac{a_r}{r^\alpha}\left(1-\left(1-\tfrac{1}{r^\beta}\right)\right)^{\frac{s-\alpha}{\beta}} &= \sum_{k=0}^{\infty}P_k(\tfrac{s-\alpha}{\beta}+1)\sum_{r=1}^{\infty}\tfrac{a_r}{r^\alpha}\left(1-\tfrac{1}{r^\beta}\right)^k\\ &= \sum_{k=0}^{\infty}P_k(\tfrac{s-\alpha}{\beta}+1)\sum_{j=0}^{k}(-1)^j\tbinom{k}{j}\sum_{r=1}^{\infty}\tfrac{a_r}{r^{\alpha+\beta j}}\\ &= \sum_{k=0}^{\infty}P_k(\tfrac{s-\alpha}{\beta}+1)\sum_{j=0}^{k}(-1)^j\tbinom{k}{j}\ln\Big(\sum_{n=1}^{\infty}\tfrac{(-1)^{n-1}}{n^{\alpha+\beta j}}\Big)\end{aligned}$$

which coincide with (3.2) and (3.3), obtained with the Euler product formula for $\Re(s)>1$. (3.2) with (3.3), is the new formula possibly representing the logarithm of the Zeta function in terms of the two parameters Pochhammer polynomials. To the best of our knowledge the above representation is new and it is our aim to carry out some numerical investigations in the sequel in order to support its validity also in some compact subset of the critical strip.

We now investigate the representation of the derivative of $\ln[(s-1)\zeta(s)]$:

$$\frac{d}{ds}\ln\left[(s-1)\,\zeta(s)\right]=\frac{1}{s-1}+\frac{\zeta'(s)}{\zeta(s)} \tag{3.4}$$

Then with $\zeta(s)=\prod\limits_{p\,\text{prime}}\frac{1}{1-p^{-s}}$ we obtain ($\Re(s)>1$):

$$\begin{aligned}\tfrac{\zeta'(s)}{\zeta(s)} &= -\sum_p\tfrac{d}{ds}\ln\left(1-p^{-s}\right)=-\sum_p\tfrac{1}{1-p^{-s}}\tfrac{d}{ds}\left(1-e^{-s\ln p}\right)\\ &= -\sum_p\tfrac{p^{-s}}{1-p^{-s}}\ln p=-\sum_p\ln p\sum_{q=1}^{\infty}\tfrac{1}{p^{sq}}\end{aligned}$$

Introducing as above the Pochhammer polynomials we obtain further:

$$\begin{aligned}\tfrac{\zeta'(s)}{\zeta(s)} &= -\sum_p\ln p\sum_{q=1}^{\infty}\tfrac{1}{p^{q\alpha}}\Big(1-\Big(1-\tfrac{1}{p^{q\beta}}\Big)\Big)^{\frac{s-\alpha}{\beta}}\\ &= -\sum_p\ln p\sum_{k=0}^{\infty}P_k(\tfrac{s-\alpha}{\beta}+1)\sum_{j=0}^{k}(-1)^j\tbinom{k}{j}\sum_{q=1}^{\infty}\tfrac{1}{p^{q(\alpha+\beta j)}}\\ &= \sum_{k=0}^{\infty}P_k(\tfrac{s-\alpha}{\beta}+1)\sum_{j=0}^{k}(-1)^j\tbinom{k}{j}\sum_{q=1}^{\infty}\Big(-\sum_p\tfrac{1}{p^{q(\alpha+\beta j)}}\ln p\Big)\\ &= \sum_{k=0}^{\infty}P_k(\tfrac{s-\alpha}{\beta}+1)\sum_{j=0}^{k}(-1)^j\tbinom{k}{j}\tfrac{\partial}{\partial\alpha}\Big(\sum_{q=1}^{\infty}\tfrac{1}{q}\sum_p\tfrac{1}{p^{q(\alpha+\beta j)}}\Big)\\ &= \sum_{k=0}^{\infty}P_k(\tfrac{s-\alpha}{\beta}+1)\sum_{j=0}^{k}(-1)^j\tbinom{k}{j}\tfrac{\partial}{\partial\alpha}\Big(-\sum_p\ln\Big(1-\tfrac{1}{p^{\alpha+\beta j}}\Big)\Big)\\ &= \sum_{k=0}^{\infty}P_k(\tfrac{s-\alpha}{\beta}+1)\sum_{j=0}^{k}(-1)^j\tbinom{k}{j}\tfrac{\partial}{\partial\alpha}\ln\Big(\prod_p\tfrac{1}{1-p^{-(\alpha+\beta j)}}\Big)\\ &= \sum_{k=0}^{\infty}P_k(\tfrac{s-\alpha}{\beta}+1)\sum_{j=0}^{k}(-1)^j\tbinom{k}{j}\tfrac{\partial}{\partial\alpha}\ln\left(\zeta(\alpha+\beta j)\right)\end{aligned}$$

For $\frac{1}{s-1}$, using $\frac{1}{s-1}=\int_0^\infty e^{-\lambda(s-1)}d\lambda$ we have similarly:

$$\begin{aligned}
\tfrac{1}{s-1} &= \int_0^\infty e^{\lambda}\tfrac{1}{e^{\lambda s}}d\lambda = \int_0^\infty \tfrac{e^{\lambda}}{e^{\lambda\alpha}}\big(1-\big(1-\tfrac{1}{e^{\lambda\beta}}\big)\big)^{\frac{s-\alpha}{\beta}}d\lambda \\
&= \int_0^\infty e^{\lambda}\sum_{k=0}^{\infty} P_k(\tfrac{s-\alpha}{\beta}+1)\sum_{j=0}^{k}(-1)^j\tbinom{k}{j}\tfrac{1}{e^{\lambda(\alpha+\beta j)}}d\lambda \\
&= \sum_{k=0}^{\infty} P_k(\tfrac{s-\alpha}{\beta}+1)\sum_{j=0}^{k}(-1)^j\tbinom{k}{j}\int_0^\infty e^{-\lambda(\alpha+\beta j-1)}d\lambda \\
&= \sum_{k=0}^{\infty} P_k(\tfrac{s-\alpha}{\beta}+1)\sum_{j=0}^{k}(-1)^j\tbinom{k}{j}\tfrac{1}{\alpha+\beta j-1} \\
&= \sum_{k=0}^{\infty} P_k(\tfrac{s-\alpha}{\beta}+1)\sum_{j=0}^{k}(-1)^j\tbinom{k}{j}\tfrac{\partial}{\partial\alpha}\ln(\alpha+\beta j-1)
\end{aligned}$$

Thus, along these lines we obtain:

$$\frac{d}{ds}\ln\left[(s-1)\,\zeta(s)\right] = \sum_{k=0}^{\infty}\hat{d}_k P_k\Big(\frac{s-\alpha}{\beta}+1\Big) \tag{3.5}$$

where:

$$\hat{d}_k = \sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{\partial}{\partial\alpha}\ln\left[(\alpha+\beta j-1)\,\zeta(\alpha+\beta j)\right] \tag{3.6}$$

From the formula (7) in [11], where ρ represents a non-trivial zero of the Zeta function, i.e.:

$$\begin{aligned}
\tfrac{1}{s-1}+\tfrac{\zeta'(s)}{\zeta(s)} &= \tfrac{1}{s-1}-\tfrac{s}{s-1}+\sum_\rho\tfrac{1}{\rho}+\sum_\rho\tfrac{1}{s-\rho}-\sum_{n=1}^{\infty}\tfrac{1}{2n}+\sum_{n=1}^{\infty}\tfrac{1}{s+2n}+\tfrac{\zeta'(0)}{\zeta(0)} \\
&= \tfrac{\zeta'(0)}{\zeta(0)}-1+\sum_\rho\tfrac{1}{\rho}-\sum_{n=1}^{\infty}\tfrac{1}{2n}+\sum_\rho\tfrac{1}{s-\rho}+\sum_{n=1}^{\infty}\tfrac{1}{s+2n}
\end{aligned}$$

Setting $C=\frac{\zeta'(0)}{\zeta(0)}-1$, this equation applied to $s=\alpha+\beta j$ in (3.6) gives:

$$\begin{aligned}
\hat{d}_k &= \sum_{j=0}^{k}(-1)^j\tbinom{k}{j}\Big(C+\int_0^\infty\Big(\sum_\rho e^{-\lambda(\alpha+\beta j-\rho)}+e^{-\lambda\rho} \\
&\quad+\sum_{n=1}^{\infty}e^{-\lambda(\alpha+\beta j+2n)}-e^{-\lambda 2n}\Big)d\lambda\Big) \\
&= \int_0^\infty\sum_\rho\Big(e^{-\lambda(\alpha-\rho)}\big(1-\tfrac{1}{e^{\lambda\beta}}\big)^k+e^{-\lambda\rho}\,\big(1-\tfrac{1}{e^{\lambda\beta}}\big)^k\delta_{k,0}\Big)\,d\lambda \\
&\quad+\int_0^\infty\Big(\sum_{n=1}^{\infty}e^{-\lambda(\alpha+2n)}\big(1-\tfrac{1}{e^{\lambda\beta}}\big)^k-e^{-\lambda 2n}\big(1-\tfrac{1}{e^{\lambda\beta}}\big)^k\delta_{k,0}\Big)d\lambda
\end{aligned}$$

We consider only $k>0$. Now we make the variable change $e^{-\lambda\beta}=x$ and finally we obtain:

$$\begin{aligned}
\hat{d}_k &= \tfrac{1}{\beta}\Big(\int_0^1(1-x)^{k+1-1}\sum_\rho x^{\frac{\alpha-\rho}{\beta}-1}dx+\int_0^1(1-x)^{k+1-1}\sum_{n=1}^{\infty}x^{\frac{\alpha+2n}{\beta}-1}dx\Big) \\
&= \tfrac{1}{\beta}\Big(\sum_\rho B(\tfrac{\alpha-\rho}{\beta},k+1)+\sum_{n=1}^{\infty}B(\tfrac{\alpha+2n}{\beta},k+1)\Big)
\end{aligned}$$

where $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Beta function. Thus for large k we can write:

$$\hat{d}_k = \frac{1}{\beta}\sum_\rho\Gamma\Big(\frac{\alpha-\rho}{\beta}\Big)k^{-\frac{\alpha-\rho}{\beta}}+\frac{1}{\beta}\sum_{n=1}^{\infty}\Gamma\Big(\frac{\alpha+2n}{\beta}\Big)k^{-\frac{\alpha+2n}{\beta}} \tag{3.7}$$

For the critical function (see the definition in [7] corresponding to $\Re(s) = \sigma$ we have an analogous expression to the Baez-Duarte formula for the c_k appearing in the expansion of $\zeta(s)^{-1}$ [2, 4]:

$$k^{\frac{\alpha-\sigma}{\beta}} \hat{d}_k = \frac{1}{\beta} \sum_{\rho} \Gamma\Big(\frac{\alpha-\rho}{\beta}\Big) k^{\frac{\rho-\sigma}{\beta}} + \frac{1}{\beta} \sum_{n=1}^{\infty} \Gamma\Big(\frac{\alpha+2n}{\beta}\Big) k^{-\frac{2n+\sigma}{\beta}} =: \psi_1(k) \tag{3.8}$$

On the other hand we can express $\hat{d}_k$ and then the critical function with a second formula:

$$\hat{d}_k = \frac{1}{\beta} \Gamma\Big(\frac{\alpha-1}{\beta}\Big) k^{-\frac{\alpha-1}{\beta}} - \sum_{p \text{ prime}} \ln p \sum_{q=1}^{\infty} \frac{1}{p^{\alpha q}} \Big(1 - \frac{1}{p^{\beta q}}\Big)^k \tag{3.9}$$

$$k^{\frac{\alpha-\sigma}{\beta}} \hat{d}_k = \frac{1}{\beta} \Gamma\Big(\frac{\alpha-1}{\beta}\Big) k^{\frac{1-\sigma}{\beta}} - k^{\frac{\alpha-\sigma}{\beta}} \sum_{p \text{ prime}} \ln p \sum_{q=1}^{\infty} \frac{1}{p^{\alpha q}} \Big(1 - \frac{1}{p^{\beta q}}\Big)^k =: \psi_2(k) \tag{3.10}$$

In fact (see above) the Pochhammer expansion for $\frac{1}{s-1}$ is:

$$\frac{1}{s-1} = \sum_{k=0}^{\infty} s_k P_k\Big(\frac{s-\alpha}{\beta} + 1\Big)$$

where

$$s_k = \int_0^{\infty} e^{-\lambda(\alpha-1)} \big(1 - e^{-\lambda\beta}\big)^k d\lambda$$

which for large k behaves as $\frac{1}{\beta}\Gamma(\frac{\alpha-1}{\beta})k^{-\frac{\alpha-1}{\beta}}$. Indeed with the substitution $e^{-\lambda\beta} = x$ we obtain:

$$s_k = \frac{1}{\beta} \int_0^1 x^{\frac{\alpha-1}{\beta}-1} (1-x)^k dx = \frac{1}{\beta} \int_0^1 x^{\frac{\alpha-1}{\beta}-1} (1-x)^{k+1-1} dx = \frac{1}{\beta} B\Big(\frac{\alpha-1}{\beta}, k+1\Big)$$

It is interesting to note that one can express the critical function in terms of the zeros of the Zeta function (3.8) or in terms of the primes (3.10). We will investigate numerically these two functions for the case $\alpha = \frac{9}{2}$, $\beta = 4$, $\sigma = \frac{1}{2}$, although we derived (3.8) only for $\sigma > 1$.

4. Numerical experiments

As a test of the goodness of (3.2) we draw in Figure 3 the plots of the function $\ln\big[(1 - 2^{1-\sigma})\zeta(\sigma)\big]$ and of its polynomial representation in the interval $\sigma \in [-1, 1[$. Figure 3 shows a good match between them also in the "critical real interval" $[0, 1]$. We set $\alpha = 2, \beta = 2$ and $k = 50$.

In the next two figures we present the results of the numerical experiment performed on our representation (3.5) for the case $\alpha = \frac{9}{2}$ and $\beta = 4$. Using formulae (3.8) and (3.10), we calculated the critical functions ψ_1 and ψ_2 for $\Re(z) = \sigma = \frac{1}{2}$. In our calculations we considered only the first 10 non-trivial zeros of the Zeta function, the first 20 trivial ones and the first 5'000 primes. For comparison's purpose we also did the calculations with 2'000 primes. Furthermore using the usual

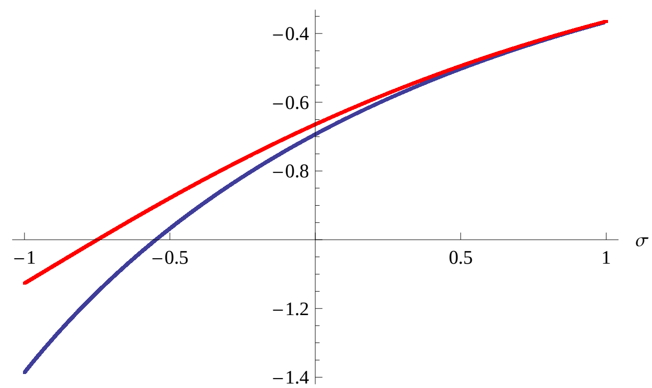

FIGURE 3. The function $\ln\left[(1-2^{1-\sigma})\zeta(\sigma)\right]$ [blue] and its polynomial representation [red]

substitution $x = \log k$, ψ_1 and ψ_2 become:

$$\psi_1(x) = \frac{\sum_{j=1}^{10}\Gamma\left(1-\frac{it_j}{4}\right)e^{\frac{xit_j}{4}} + \sum_{j=1}^{10}\Gamma\left(1+\frac{it_j}{4}\right)e^{-\frac{xit_j}{4}} + \sum_{n=1}^{20}\Gamma\left(\frac{1}{2}n+\frac{9}{8}\right)e^{-x(\frac{1}{2}n+\frac{1}{8})}}{4}$$

$$\psi_2(x) = \frac{1}{4}\Gamma\left(\frac{7}{8}\right)e^{\frac{x}{8}} - e^x \sum_{\substack{5000\\ \text{primes}}} \ln p \sum_{q=1}^{50} p^{-\frac{9}{2}q} e^{-\frac{e^x}{p^{4q}}}$$

where t_j is the imaginary part of the *j-th* non-trivial zero.

We argue ψ_2 should approach ψ_1. The convergence is surprising. The computations presented in Figure 4 and Figure 5 indicate that the qualitative and quantitative agreement between the two functions is very good in the range $2.5 \le x \le 33$ $(15 \le k \le 2.14644 \times 10^{14})$.

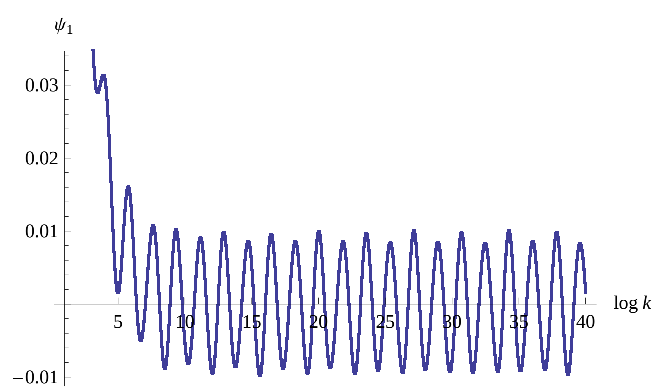

FIGURE 4. The critical function calculated with the zeros of the Zeta function (ψ_1), using the first 10 non-trivial zeros and the first 20 trivial ones

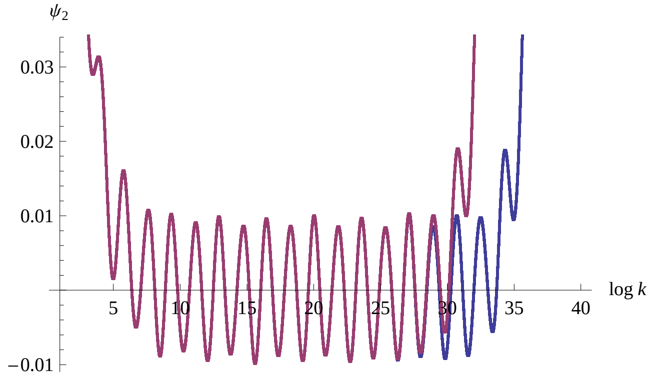

FIGURE 5. The critical function calculated with the primes (ψ_2): 2000 primes [red] and 5000 primes [blue]

Remark 4.1. We observe that as the number of primes increases from 2'000 to 5'000 ψ_2 becomes identical to ψ_1 for greater values of k. So we suspect that as the number of primes increases, ψ_1 and ψ_2 would coincide for larger and larger values of k. So there is some evidence that the two functions represent the same mathematical object. This fact, which to the best of our knowledge is new, should deserve further studies.

It is interesting to study the single contribution of a prime to the critical function ψ_2. In Figure 6 we computed the contributions of the 10th prime ($p = 29$), of the 50th prime ($p = 229$) and of the 100th prime ($p = 541$), all the calculations were performed until $q = 100$. The computations indicate that not only the contributions decrease with increasing p but also that large primes give in fact a contribution only at large values of k.

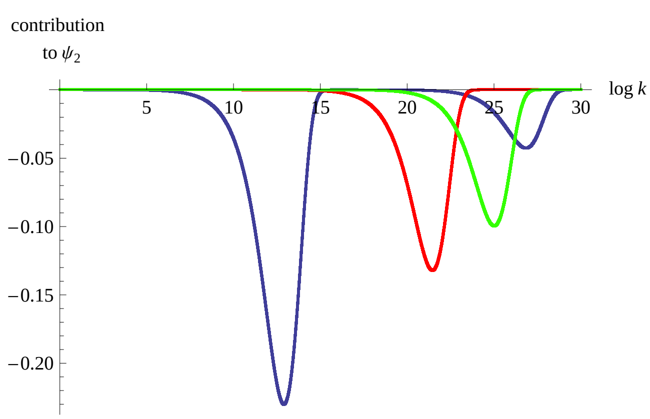

FIGURE 6. The contribution to the critical function ψ_2 of the primes $p = 29$ [blue], $p = 229$ [red] and $p = 541$ [green]

Remark 4.2. A "verification" for the truth of the RH using the representation of the function (3.4) by means of the Pochhammer polynomials may be given as follows.

Assume that $\hat{d}_k$ (either with the primes or with the zeros of the Zeta function) decays as $\hat{d}_k < \frac{D}{k^\gamma}$ with $\gamma = \frac{\alpha-1/2}{\beta}$ and some constant D; in fact this assumption is equivalent to the RH (see [4] and [6]). Then we have:

$$\left|\frac{d}{ds}\ln\left[(s-1)\,\zeta(s)\right]\right| < \left|\sum_{k=1}^{\infty} C\frac{1}{k^{\frac{\sigma-\alpha}{\beta}+1}}\frac{1}{k^{\frac{\alpha-1/2}{\beta}}}\right| < C\zeta\Big(1+\frac{\sigma-1/2}{\beta}\Big) \tag{4.1}$$

So the function would be bounded for $\sigma > \frac{1}{2}$ and there would be no zero with real part greater then $\frac{1}{2}$. In the same way the critical function ψ should behaves like:

$$\psi(\sigma) = k^{\frac{\alpha-\sigma}{\beta}} d_k < \frac{D}{k^{\frac{\sigma-1/2}{\beta}}}$$

For $\sigma = \frac{1}{2}$ we have no criteria but it seems (Figure 4) that the critical function $\psi(\frac{1}{2})$ is also bounded. We verified numerically the bound given by (4.1) at $\sigma = 0.6, 0.55, 0.525$ where we found that D is about 9.5.

Remark 4.3. Now, suppose that $\psi(\sigma')$ is bounded for some $\sigma' > \frac{1}{2}$, then since

$$\psi(\sigma) = \psi(\sigma')k^{\frac{\sigma'-\sigma}{\beta}}$$

this would indicate that if there is no zero at σ' then there is also no zero at σ. Thus it would be important to study ψ for example in the region $\sigma > 1$ where it is known that there are no zeros and where the primes (ψ_2) as well as the zeros (ψ_1) can be used.

5. Conclusions

In this work we have found some new representations of functions related to the Riemann Zeta function in terms of the Pochhammer polynomials, i.e. for the Zeta function via the alternating series, for $(1-2^{1-s})\zeta(s)$, for $\ln\left[(1-2^{1-s})\zeta(s)\right]$ and for the derivative of $\ln\left[(s-1)\zeta(s)\right]$.

(1) A numerical experiment for the first function give satisfactory results both for the real part as well for the imaginary part even on the critical line $\Re(s) = \frac{1}{2}$ (we have used the values $\alpha = \frac{1}{2}$, $\beta = i$ and t up to $t = \Im(s) < 35$).
(2) In a formal limit of our representation (2.6) for the special case $\alpha = \beta = 2$ we obtain Maslanka's representation of $(s-1)\zeta(s)$.
(3) For the expansion of the derivative of the function $\ln\left[(s-1)\zeta(s)\right]$ in terms of the Pochhammer polynomials $P_k(s)$ we have found two expressions (ψ_1 and ψ_2) for the so called critical function: ψ_1 in terms of the trivial as well as the non-trivial zeros and ψ_2 in terms of the primes. We have then carried out a numerical experiment which gives a very satisfactory agreements between the two functions, which up to very high values of k remain bounded. The existence of absolute upper bounds for the critical functions at k-infinity may be considered as being equivalent to the truth of the RH.
(4) The "equality" of ψ_1 and ψ_2 in the numerical context is intriguing because we have found a mathematical object related to the Zeta function and representable by means of the infinity of the zeros of Zeta as well as the infinity of the primes.

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