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DECOMPOSABILITY OF EXTENSION RINGS

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Abstract. Skew polynomial rings have invited attention of mathematicians and various properties of these rings have been discussed. The nature of ideals (in particular prime ideals, minimal prime ideals, associated prime ideals), primary decomposition and Krull dimension have been investigated in certain cases.

This article concerns transparent (decomposable) rings. Recall that a ring R is said to be a *Transparent ring* if in R there exist irreducible ideals I_j, $1 \leq j \leq n$ such that $\cap_{j=1}^{n} I_j = 0$ and each R/I_j has a right Artinian quotient ring.

Now let R be a ring, which is an order in an Artinian ring S. Let σ and τ be automorphisms of R and δ be a (σ,τ)-derivation of R; i.e. $\delta : R \to R$ is an additive mapping satisfying $\delta(ab) = \sigma(a)\delta(b) + \delta(a)\tau(b)$ for all a, $b \in R$. We define an extension of R, namely $R[x,\sigma,\tau,\delta] = \{f = \sum_{i=0}^{n} x^i a_i, a_i \in R\}$, subject to the relation $ax = x\sigma(\tau(a)) + \delta(a)$ for all $a \in R$.

We show that if R is a commutative Noetherian $\mathbb{Q}$-algebra, σ and τ as usual, then there exists an integer $m \geq 1$ such that the extension ring $R[x,\alpha,\beta,\vartheta]$ is a *Transparent ring*, where $\alpha = \sigma^m$, $\beta = \tau^m$ and ϑ is an (α,β)-derivation of R with $\alpha(\vartheta(a)) = \vartheta(\alpha(a))$, and $\beta(\vartheta(a)) = \vartheta(\beta(a))$, for all $a \in R$.

1. Introduction

Throughout this article R is an associative ring with identity and any R-module is unitary. Spec(R) denotes the set of prime ideals of R. Min.Spec(R) denotes the set of minimal prime ideals of R. The set of associated prime ideals of R (viewed as a module over itself) is denoted by Ass(R). For a subset U of an R-module M, the annihilator of U is denoted by Ann(U). Now let R be a Noetherian ring. For any uniform R-module K, the unique associated prime of K (known as assassinator of K) is denoted by Assas(K). C(0) denotes the set of regular elements of R. C(I) denotes the set of elements of R regular modulo I, where I is an ideal of R. N(R) denotes the prime radical of R. $\mid M \mid$r denotes the right Krull dimension of a right R-module M. For further details on Krull dimension, the reader is referred to [10]. Let I and J be any two ideals of a ring R. Then $I \subset$ J means that I is strictly

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contained in J. The field of rational numbers, the ring of integers and the set of positive integers are denoted by $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ respectively, unless otherwise stated.

Let σ and τ be automorphisms of a ring R and δ be a (σ,τ)-derivation of R; i.e. $\delta : R \to R$ is an additive mapping satisfying $\delta(ab) = \sigma(a)\delta(b) + \delta(a)\tau(b)$.
For example let σ and τ be automorphism of a ring R and $\delta : R \to R$ any map. Let $\phi : R \to M_2(R)$ be defined by

$$\phi(r) = \begin{pmatrix} \tau(r) & 0 \\ \delta(r) & \sigma(r) \end{pmatrix}, \text{ for all } r \in R.$$

Then δ is a right (σ,τ)-derivation of R.

We define an extension of R, namely $R[x,\sigma,\tau,\delta] = \{f = \sum_{i=0}^{n} x^i a_i, a_i \in R\}$, subject to the relation $ax = x\sigma(\tau(a)) + \delta(a)$ for all $a \in R$. Denote $R[x,\sigma,\tau,\delta]$ by $E(R)$. In case τ is the identity map, we denote $R[x,\sigma,\delta]$ by $O(R)$. In case τ is the identity map and δ is the zero map, we denote $R[x,\sigma]$ by $S(R)$. In case σ and τ are the identity maps, we denote $R[x,\delta]$ by $D(R)$. Note that δ in this case is just a derivation. We denote the skew Laurent polynomial ring $R[x,x^{-1},\sigma]$ by L(R).

For more details on Ore extensions (skew polynomial rings), we refer the reader to Chapter (1) of [8]. Notion of the quotient rings and contractions and extensions of ideals appear in Chapter (9) of [8].

The classical study of any commutative Noetherian ring is done by studying its primary decomposition. Further there are other structural properties of rings, for example the existence of quotient rings or more particularly the existence of Artinian quotient rings etc. which can be nicely tied to primary decomposition of a Noetherian ring.

The first important result in the theory of non commutative Noetherian rings was proved in 1958 (Goldie's Theorem) which gives an analog of field of fractions for factor rings R/P, where R is a Noetherian ring and P is a prime ideal of R. In 1959 the one sided version was proved by Goldie, Lesieur-Croisot (Theorem (5.12) of [8])and in 1960 Goldie generalized the result for semiprime rings (Theorem (5.10) of [8]).

In [5] it is shown that if R has characteristic zero and it is embeddable in a right Artinian ring, then the differential operator ring $R[x,\delta]$ embeds in a right Artinian ring, where δ is a derivation of R. It is also shown in [5] that if R is a commutative Noetherian ring and σ is an automorphism of R, then the skew-polynomial ring $R[x,\sigma]$ embeds in an Artinian ring.

In this paper the above mentioned properties have been studied with emphasis on primary decomposition of the Ore extension E(R), where R is a commutative Noetherian $\mathbb{Q}$-algebra, where σ and τ are automorphisms of R and δ is a (σ,τ)-derivation of R.

A non commutative analogue of associated prime ideals of a Noetherian ring has also been also discussed. We would like to note that a considerable work has been done in the investigation of prime ideals (in particular minimal prime ideals and associated prime ideals) of skew polynomial rings (K. R. Goodearl and E. S. Letzter [9], C. Faith [6], S. Annin [1], Leroy and Matczuk [12], Nordstrom [14] and Bhat [4]).

In section (4) of [9] Goodearl and Letzter have proved that if R is a Noetherian ring, then for each prime ideal P of O(R), the prime ideals of R minimal over $P \cap R$ are contained within a single σ-orbit of Spec(R).

The author has proved in Theorem (2.4) of [4] that if σ is an automorphism of a Noetherian ring R and K(R) is any of S(R) or L(R), then $P \in Ass(K(R))$ if and only if there exists $U \in Ass(R)$ such that $K(P \cap R) = P$ and $P \cap R = \cap_{i=0}^{m} \sigma^i(U)$, where $m \geq 1$ is an integer such that $\sigma^m(V) = V$ for all $V \in Ass(R)$. (Same result has been proved for minimal prime ideal case).

Carl Faith has proved in [6] that if R is a commutative ring, then the associated prime ideals of the usual polynomial ring R[x] (viewed as a module over itself) are precisely the ideals of the form P[x], where P is an associated prime ideal of R.

S. Annin has proved in Theorem (2.2) of [1] that if R is a ring and M be a right R-module. If σ is an endomorphism of R and S = $R[x, \sigma]$ and M_R is σ-compatible, then $Ass(M[x]_S) = \{P[x]$ such that $P \in Ass(M_R)\}$.

In [12], Leroy and Matczuk have investigated the relationship between the associated prime ideals of an R-module M_R and that of the induced S - module M_S, where S = $R[x, \sigma]$ (σ an automorphism of a ring R). They have proved the following:

Theorem (5.7) of [12]: Suppose M_R contains enough prime submodules and let $Q \in Ass(M_S)$. If for every $P \in Ass(M_R)$, $\sigma(P) = P$, then Q = PS for some $P \in Ass(M_R)$.

In Theorem (1.2) of [14] Nordstrom has proved that if R is a ring with identity and σ is a surjective endomorphism of R, then for any right R-module M, $Ass(M[x, \sigma]) = \{I[x, \sigma],\ I \in \sigma - Ass(M)\}$. In Corollary (1.5) of [14] it has been proved that if R is Noetherian and σ is an automorphism of R, then $Ass(M[x, \sigma]_S)$ = $\{P_\sigma[x, \sigma],\ P \in Ass(M)\}$, where $P_\sigma = \cap_{i \in N} \sigma^{-i}(P)$ and S = $R[x, \sigma]$.

The above discussion leads to a stronger type of primary decomposition of a Noetherian ring. We call a Noetherian ring with such a decomposition a *Transparent ring*.

Before we give the definition of a *Transparent ring*, we need the following:

Definition 1.1. A ring R is said to be an irreducible ring if the intersection of any two non-zero ideals of R is non-zero. An ideal I of R is called irreducible if $I = J \cap K$ implies that either $J = I$ or $K = I$. Note that if I is an irreducible ideal of R, then R/I is an irreducible ring.

Proposition 1.2. *Let R be a Noetherian ring. Then there exist irreducible ideals* I_j, $1 \leq j \leq n$ *of R such that* $\cap_{j=1}^{n} I_j = 0$.

Proof. The proof is obvious and we leave the details to the reader. □

Definition (A): A Noetherian ring R is said to be a *Transparent ring* if there exist irreducible ideals I_j, $1 \leq j \leq n$ such that $\cap_{j=1}^{n} I_j = 0$ and each R/I_j has a right Artinian quotient ring.

It can be easily seen that an integral domain is a *Transparent ring*, a commutative Noetherian ring is a *Transparent ring* and so is a Noetherian ring having an Artinian quotient ring. A fully bounded Noetherian ring is also a *Transparent ring*.

This type of decomposition was actually introduced by the author in [2]. Such a ring was called a *decomposable ring*, but in order to distinguish between one more definition of a *decomposable ring* given below and pointed out by the referee of one of the papers of the author, we now call such a ring a *Transparent ring*.

Decomposable ring (Hazewinkel and Krichenko[11]**)** Let R be a ring. An R-module M is said to be decomposable if $M \simeq M_1 \oplus M_2$ of non zero R-modules M_1 and M_2. A ring R is called a *decomposable ring* if it is a direct sum of two rings.

Now there arises a natural question: If R is a *Transparent ring*; σ, τ and δ are as usual. Is E(R) a *Transparent ring*? We have not been able to answer this question in general, however, in commutative case we have the following:
If R is a commutative Noetherian $\mathbb{Q}$-algebra; σ and τ are automorphisms of R, then there exists an integer $m \geq 1$ such that the extension ring $R[x, \alpha, \beta, \vartheta]$ is decomposable, where $\alpha = \sigma^m$, $\beta = \tau^m$ and ϑ is an (α, β)-derivation of R with $\alpha(\vartheta(a)) = \vartheta(\alpha(a))$, and $\beta(\vartheta(a)) = \vartheta(\beta(a))$, for all $a \in R$. This is proved in Theorem (3.12).

Before proving the main result, we recall that if R is a ring, which is an order in an Artinian ring S. If σ and τ are automorphisms of R and δ is a (σ, τ)-derivation of R, then σ and τ can be extended to automorphisms α and β (say) of S and δ can be extended to an (α, β)-derivation (say) ρ of S. This has been proved in Proposition (2.1) of [3]. In Theorem (2.11) of [3] it has been proved that that E(R) is an order in $E(S)_L$, where $E(S) = S[x, \alpha, \beta, \rho]$ and L is the set of monic polynomials of E(S).

2. Preliminaries

Recall that the skew power series ring $R[[t, \sigma, \tau]]$ is as a set the power series ring $R[[t]]$ in which multiplication is subject to the relation $ax = x\sigma(\tau(a))$, for all $a \in R$. Denote $R[[t, \sigma, \tau]]$ by T.

Definition 2.1. Let R be a ring. Let σ and τ be automorphisms of R and δ be a (σ, τ)-derivation of R. Then $R[x, \sigma, \tau, \delta] = \{f = \sum_{i=0}^{n} x^i a_i, a_i \in R\}$, subject to the relation $ax = x\sigma(\tau(a)) + \delta(a)$ for all $a \in R$.

Remark 2.2. If $\sigma(\delta(a)) = \delta(\sigma(a))$ and $\tau(\delta(a)) = \delta(\tau(a))$, for all $a \in R$, then σ and τ can be extended to an automorphisms of E(R) such that $\sigma(x) = x$ and $\tau(x) = x$ and δ can be extended to a (σ, τ)-derivation of E(R) such that $\delta(x) = 0$, that is $\sigma(xa) = x\sigma(a)$, $\tau(xa) = x\tau(a)$ and $\delta(xa) = x\delta(a)$.

Lemma 2.3. *Let R be a Noetherian $\mathbb{Q}$-algebra; σ and τ automorphisms of R and δ a (σ, τ)-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$, and $\tau(\delta(a)) = \delta(\tau(a))$ for all $a \in R$. Then $e^{t\delta} = 1 + t\delta + (t^2/2!)\delta^2 + ...$is an automorphism of T.*

Proof. The proof is on the same lines as in [16] and in non-commutative case, it is similar to the sketch of the proof provided in [5]. □

Remark 2.4. Let R be a Noetherian ring; σ, τ and δ as usual. Let I be an ideal of R such that $\sigma(I) = I$ and $\tau(I) = I$. Then $IT = \{b_0 + tb_1 + t^2 b_2 + ..., b_i \in I\}$. We denote it by $I[[t, \sigma, \tau]]$.

Lemma 2.5. *Let R be a Noetherian $\mathbb{Q}$-algebra; σ, τ and δ as usual such that $\sigma(\delta(a)) = \delta(\sigma(a))$ and $\tau(\delta(a)) = \delta(\tau(a))$, for $a \in R$. Then an ideal I of R is δ-invariant if and only if IT is $e^{t\delta}$-invariant.*

Proof. Let IT be $e^{t\delta}$-invariant. Let $a \in I$. Then $a \in IT$. So $e^{t\delta}(a) \in IT$; i.e. $a + t\delta(a) + (t^2/2!)\delta^2(a) + ... \in IT$, which implies that $\delta(a) \in I$.
Conversely suppose that $\delta(I) \subseteq I$ and let $f = \sum_{j=0}^{\infty} t^j a_j \in IT$. Then $e^{t\delta}(f) =$

$f + t\delta(f) + (t^2\delta^2/2!)(f) + ... = \sum_{j=0}^{\infty} t^j a_j + t(\sum_{j=0}^{\infty} t^j \delta(a_j) + ... \in IT$, as $\delta(a_i) \in I$. Therefore $e^{t\delta}(IT) \subseteq IT$. Replacing $e^{t\delta}$ by $e^{-t\delta}$, we get that $e^{t\delta}(IT) = IT$. □

Lemma 2.6. *Let R be a Noetherian ring; σ, τ and δ as usual such that $\sigma(\delta(a)) = \delta(\sigma(a))$ and $\tau(\delta(a)) = \delta(\tau(a))$, for $a \in R$ and T be as usual. Then:*

1. *$A \in Ass(R)$ implies that $AT \in Ass(T)$.*
2. *$P \in Ass(T)$ implies that $P \cap R \in Ass(R)$ and $P = (P \cap R)T$.*

Proof. **(1)** Let A = Ann(cR) = Assas(cR), $c \in R$. Then it can be seen that $AT \in Spec(T)$ and AT = Ann(cT) = Assas(cT). Therefore $AT \in Ass(T)$

(2) Let f $= a_0 + ta_1 + t^2a_2 + ... \in T$ be such that P = Ann(fT) = Assas(fT). Now $a_iR(P \cap R) = 0$ for all i. Choose $a_n \neq 0$ from coefficients of f. Let $U = Ann(a_nR)$. Now $U = Ann(a_nrR) = Assas(a_nrR)$, $r \in R$ such that $a_nr \neq 0$. Now it is easy to see that $UT = Ann(a_nrRT) = Assas(a^nrRT)$. Now it can be seen that $U = P \cap R$. Therefore $P \cap R \in Ass(R)$ and $P = (P \cap R)T$. □

Lemma 2.7. *Let R be a ring; σ, τ and δ as usual such that $\sigma(\delta(a)) = \delta(\sigma(a))$ and $\tau(\delta(a)) = \delta(\tau(a))$, for $a \in R$ and T be as usual. Then:*

1. *$A \in Min.Spec(R)$ implies that $AT \in Min.Spec(T)$*
2. *$P \in Min.Spec(T)$ implies that $P \cap R \in Min.Spec(R)$ and $P = (P \cap R)T$.*

Proof. The proof follows on the same lines as in Proposition (2.5) of [4]. □

In Lemma (3.4) of [7], Gabriel proved that if R is a Noetherian $\mathbb{Q}$-algebra and δ is a derivation of R, then $\delta(P) \subseteq P$ for all $P \in Min.Spec(R)$. In Theorem (1) of [16], Seidenberg proved that if R is a commutative Noetherian $\mathbb{Q}$-algebra and δ is a derivation of R, then $\delta(P) \subseteq P$ for all $P \in Ass(R)$. We generalize these results and prove them in one go. Towards this we have the following:

Lemma 2.8. *Let R be a Noetherian $\mathbb{Q}$-algebra; σ, τ and δ as usual such that $\sigma(\delta(a)) = \delta(\sigma(a))$ and $\tau(\delta(a)) = \delta(\tau(a))$, for $a \in R$. Then $P \in Ass(R) \cup Min.Spec(R)$ such that $\sigma(P) = P$ and $\tau(P) = P$ implies that $\delta(P) \subseteq P$.*

Proof. Let T be as usual. Now by Lemma (2.3) $e^{t\delta}$ is an automorphism of T. Let $P \in Ass(R) \cup MinS.pec(R)$. Then by Lemma (2.6) and Lemma (2.7) $PT \in Ass(T) \cup Min.Spec(T)$. So there exists an integer $n \geq 1$ such that $(e^{t\delta})^n(PT) = PT$; i.e. $e^{nt\delta}(PT) = PT$. But R is a $\mathbb{Q}$-algebra, therefore $e^{t\delta}(PT) = PT$, and so Lemma 2.5 implies that $\delta(P) \subseteq P$. □

Definition 2.9. A ring R is said to be an irreducible ring if the intersection of any two non-zero ideals of R is non-zero. An ideal I of R is called irreducible if $I = J \cap K$ implies that either J = I or K = I. Note that if I is an irreducible ideal of R, then R/I is an irreducible ring.

Lemma 2.10. *Let R be a Noetherian ring. Then there exist irreducible ideals I_j, $1 \leq j \leq n$ of R such that $\cap_{j=1}^{n} I_j = 0$.*

Proof. Suppose such ideals do not exist. Consider K = {Ideals J of R such that J is not intersection of irreducible ideals of R}. Now $K \neq \phi$ as $\{0\} \in K$. Now by Noetherian condition K has a maximal element (say T), and T is reducible. Let T = $U \cap V$, $T \subset U$ and $T \subset V$. Therefore U and V both are intersection of irreducible ideals of R by the maximality of T, which implies that T is an intersection of irreducible ideals of R, a contradiction. □

Lemma 2.11. *Let R be a Noetherian ring having a right Artinian quotient ring. Then R is a Transparent ring.*

Proof. Let $Q(R)$ be the right quotient ring of R . Now for any ideal J of $Q(R)$, the contraction J^c of J is an ideal of R and the extension of J^c is J; i.e. $J^{ce} = J$. For this see Proposition (9.19) of [8]. Let I_j, $1 \leq j \leq n$ be the ideals of $Q(R)$ such that $0 = \cap_{j=1}^{n} I_j$ where each $Q(R)/I_j$ is an irreducible ring. Also each $Q(R)/I_j$ is an Artinian ring. Let $I_j^c = K_j$. Then R/K_j has right Artinian quotient ring $Q(R)/I_j$ and each R/K_j is irreducible. Moreover $\cap_{j=1}^{n} K_j = 0$. Hence R is a *Transparent ring.* □

Definition 2.12. Let P be a prime ideal of a commutative ring R. Then the symbolic power of P for a positive integer n is denoted by $P^{(n)}$ and is defined as $P^{(n)} = \{a \in R$ such that there exists some $d \in R$, $d \notin P$ such that $da \in P^n\}$. Also if I is an ideal of R, define as usual $\sqrt{I} = \{a \in R$ such that $a^n \in I$ for some $n \in \mathbb{Z}$ with $n \geq 1\}$.

Lemma 2.13. *Let R be a commutative Noetherian ring, and σ an automorphism of R. If P is a prime ideal of R such that $\sigma(P) = P$, then $\sigma(P^{(n)}) = P^{(n)}$ for all integers $n \geq 1$.*

Proof. We have $\sigma(P) = P$. Let $a \in P^{(n)}$. Then there exists some $d \in R$, $d \notin P$ such that $da \in P^n$. Therefore $\sigma(da) \in \sigma(P^n)$; i.e. $\sigma(d)\sigma(a) \in (\sigma(P))^n = P^n$. Now $\sigma(d) \notin P$ implies that $\sigma(a) \in P^{(n)}$. Therefore $\sigma(P^{(n)}) \subseteq P^{(n)}$. Hence $\sigma(P^{(n)}) = P^{(n)}$. □

Lemma 2.14. *Let R be a commutative Noetherian ring; σ, τ and δ as usual. Let P be a prime ideal of R such that $\sigma(P) = P$, $\tau(P) = P$ and $\delta(P) \subseteq P$. Then $\delta(P^{(k)}) \subseteq P^{(k)}$.*

Proof. Let $a \in P^{(k)}$. Then there exists $d \notin P$ such that $da \in P^k$.
Let $da = p_1.p_2...p_t$, $p_i \in P$.

Now

$$\begin{aligned} \delta(da) &= \sigma(p_1p_2...p_{t-1})\delta(p_t) + \sigma(p_1p_2...p_{t-2})\delta(p_{t-1})\tau(p_t) + \\ &\quad ... + \sigma(p_1)\delta(p_2)\tau(p_3...p_t) + \delta(p_1)\tau(p_2...p_t) \in P^k \end{aligned} \tag{2.1}$$

as $\sigma(P) = P$, $\tau(P) = P$ and $\delta(P) \subseteq P$; i.e. $\sigma(d)\delta(a) + \delta(d)\tau(a) \in P^k$. Now $\tau(a) \in P^{(k)}$ by 2.13, and therefore $\sigma(d)\delta(a) \in P^{(k)}$, which implies that there exists $d_1 \notin P$ such that $d_1\sigma(d)\delta(a) \in P^k$ and since $d_1\sigma(d) \notin P$, we have $\delta(a) \in P^{(k)}$. □

3. Main Result

In this section we prove the main result in the form of Theorem 3.12. We begin with the following Lemma:

Lemma 3.1. *Let R be a ring which is an order in an Artinian ring S; σ, τ and δ as usual. Then σ can be extended to an automorphism (say) α of S, τ can be extended to an automorphism (say) β of S and δ can be extended to an (α, β)-derivation (say) ρ of S.*

Proof. Proposition (2.1) of [3]. □

We now state some definitions, Lemmas and analog of some results of [5], which lead us to the main result. The corresponding results and other details can be seen in [5].

Definition 3.2. Let R be a ring and U be a right Ore set in R. Let M be a right R-module. The set $T_U(M) = \{m \in M$ such that mu = 0 for some $u \in U\}$ is called the U-torsion submodule of M. M is said to be U-torsion if and only if $T_U(M) = M$ and is said to be torsion free if $T_U(M) = 0$.

Definition 3.3. Let R be a ring; σ, τ and δ as usual. Let $B = \{f \in E(R)$ such that f is monic$\}$.

Lemma 3.4. *Let R be a right Noetherian ring. Let σ, τ and δ be as usual. Let B be the set of monic polynomials of E(R). Then B is a right denominator set in E(R).*

Proof. On the same lines as in Proposition (7.9.3) of [13]. □

Lemma 3.5. *Let R be a ring. A right E(R)-module W is B-torsion if and only if every finitely generated E(R)-submodule of W is finitely generated as an R-module.*

Proof. Lemma (2.4) of [3]. □

Theorem 3.6. *Let R be a right Noetherian ring and B as usual. Then $\mid E(R)_B \mid r$ $= \mid R \mid r$, where $E(R)_B$ denotes the usual localization of E(R) at B.*

Proof. On the same lines as in Theorem (7.9.4) of [13]. □

We now state the following Lemma, the proof is left to the reader.

Lemma 3.7. *Let R be a ring which is an order in a right Artinian ring S; σ, τ and δ as usual. Then:*

(1) *Every regular element of R is regular in E(R).*
(2) *Set of regular elements of R satisfies the right Ore-condition in E(R).*
(3) *Any element of E(S) has the form $f(x).c^{-1}$ for some $f(x) \in E(R)$ and some c regular in R.*
(4) *If $g(x) = f(x).c^{-1}$ is regular in E(S), then f(x) is regular in E(R).*
(5) *Let K be the set of monic polynomials of E(S). Then every regular element of E(R) is right regular as an element of $E(S)_K$.*

Definition 3.8. Let R be a ring which has a right(respectively left) quotient ring Q(R). A multiplicative closed subset I of regular elements of R is said to be exhaustive if any $q \in Q(R)$ is such that $q = ra^{-1}$ (respectively $q = a^{-1}r$) for some $r \in R$ and some $a \in I$.

Definition 3.9. Let R be a ring. Define $M(E(R)) = \{f \in E(R)$ such that leading coefficient of f is regular in R$\}$

Lemma 3.10. *Let R be a semiprime Noetherian ring. Then M(E(R)) is an exhaustive set.*

Proof. The proof is obvious. □

Theorem 3.11. *Let R be a ring which is an order in a right Artinian ring S. Then E(R) is an order in a right Artinian ring and E(R) has an exhaustive set of elements which have leading coefficients regular in R.*

Proof. Theorem (2.11) of [3] □

We are now in a position to state and prove the main result in the form of the following Theorem:

Theorem 3.12. *Let R be a commutative Noetherian $\mathbb{Q}$-algebra, σ and τ be automorphisms of R. Then there exists an integer $m \geq 1$ such that the extension ring $R[x, \alpha, \beta, \delta]$ is a Transparent ring, where $\sigma^m = \alpha$, $\tau^m = \beta$ and δ is an (α, β)-derivation of R such that $\alpha(\delta(a)) = \delta(\alpha(a))$ and $\beta(\delta(a)) = \delta(\beta(a))$, for all $a \in R$.*

Proof. $R[x, \alpha, \beta, \delta]$ is Noetherian by Hilbert Basis Theorem, namely Theorem (1.12) of [8]. Now R is a commutative Noetherian $\mathbb{Q}$-algebra, therefore, the ideal (0) has a reduced primary decomposition. Let I_j, $1 \leq j \leq n$ be irreducible ideals of R such that $(0) = \cap_{j=1}^{n} I_j$. For this see Theorem (4) of [17]. Let $\sqrt{I_j} = P_j$, where P_j is a prime ideal belonging to I_j. Now by Theorem (23) of [17] there exists a positive integer k such that $P_j^{(k)} \subseteq I_j$, $1 \leq j \leq n$. Therefore we have $\cap_{j=1}^{n} P_j^{(k)} = 0$. Now $P_j \in Ass(R)$, $1 \leq j \leq n$ by first uniqueness Theorem. Since Ass(R) is finite, and $\psi^i(P) \in Ass(R)$ for any automorphism ψ of R, for all $i \geq 1$, there exists an integer $m \geq 1$ such that $\sigma^m(P_j) = P_j$ and $\tau^m(P_j) = P_j$. Denote σ^m by α and τ^m by β. Now $\alpha(P_j) = P_j$ and $\beta(P_j) = P_j$. Therefore $\alpha(P_j^{(k)}) = P_j^{(k)}$ and $\beta(P_j^{(k)}) = P_j^{(k)}$ by Lemma (2.13). Also $\delta(P_j) \subseteq P_j$ by Lemma 2.8 and therefore $\delta(P_j^{(k)}) \subseteq P_j^{(k)}$ by Lemma (2.14). Thus $P_j^{(k)}[x, \alpha, \beta, \delta]$ is an ideal of $R[x, \alpha, \beta, \delta]$. Now $R/P_j^{(k)}$ has no embedded primes, therefore $R/P_j^{(k)}$ has an Artinian quotient ring by Theorem (2.11) of ([15]). Now by Theorem (3.11) $R[x, \alpha, \beta, \delta]/P_j^{(k)}[x, \alpha, \beta, \delta]$ has an Artinian quotient ring. Moreover $\cap_{j=1}^{n} P_j^{(k)}[x, \alpha, \beta, \delta] = 0$, therefore Lemma (2.11) implies that $R[x, \alpha, \beta, \delta]$ is a *Transparent ring.* □

Remark 3.13.
(1) Let R be a Noetherian ring having an Artinian quotient ring. Let σ be an automorphism of R and δ be a σ-derivation of R. Then $R[x, \sigma, \delta]$ is a *Transparent ring.*
(2) Let R be a commutative Noetherian ring and σ be an automorphism of R. Then the skew polynomial ring $R[x, \sigma]$ is a *Transparent ring.*
(3) Let R be a commutative Noetherian ring and σ be an automorphism of R. Then the skew Laurent polynomial ring $R[x, x^{-1}, \sigma]$ is a *Transparent ring.*
(4) Let R be a commutative Noetherian $\mathbb{Q}$-algebra and δ be a derivation of R. Then the differential operator ring D(R) $= R[x, \delta]$ is a *Transparent ring.*

Question: If R is a commutative Noetherian $\mathbb{Q}$-algebra, σ is an automorphism of R and δ is a σ-derivation of R. Is $R[x, \sigma, \tau, \delta]$ a *Transparent ring* even if $\sigma(\delta(a)) = \delta(\sigma(a))$ and $\tau(\delta(a)) = \delta(\tau(a))$, for all $a \in R$? The main hurdle is that in such a situation $\delta(P) \subseteq P$ need not imply $\delta(P^{(k)}) \subseteq P^{(k)}$.

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