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INVERSE LIMITS OF H-CLOSED SPACES

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Abstract. The main purpose of this paper is to study the non-emptiness and H-closeness of inverse limits of H-closed spaces.

1. Introduction

An *inverse system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ [4, p. 135] over a directed set A is a function which attaches to each $a \in A$ a space X_a and to each pair $a, b \in A$ such that $a \leq b$ a mapping $p_{ab} : X_b \to X_a$ such that

$$\begin{aligned} p_{aa} &= \text{identity}, \quad a \in A, \\ p_{ab}p_{bc} &= p_{ac}, \quad a \leq b \leq c. \end{aligned}$$

The *inverse limit* $\lim \mathbf{X}$ of the inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is the set of all points $\{x_a\}$ of the Cartesian product $\Pi\{X_a : a \in A\}$ satisfying $p_{ab}(x_b) = x_a$ for every $a \leq b$.

For each inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ we define [4, Proposition 2.5.1, p.135]

$$X_{ab} = \{\{x_a\} \in \Pi\{X_a : a \in A\} : p_{ab}(x_b) = x_a, \ a \leq b\}.$$

Proposition 1. [4, Proposition 2.5.1, p.135]. *The limit of an inverse system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *of a Hausdorff spaces* X_a *is closed subset of the Cartesian product* $\Pi\{X_a : a \in A\}$.

For each inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ we define [4, Theorem 3.2.13, p.188]

$$Z_a = \{\{x_a\} \in \Pi\{X_a : a \in A\} : p_{ba}(x_a) = x_b, \ b \leq a\}$$

In [4, Theorem 3.2.13, p.188] it is used that Z_a is closed in $\Pi\{X_a : a \in A\}$. This is true if each X_a is Hausdorff.

Proposition 2. *The family* $\{Z_a : a \in A\}$ *has the finite intersection property.*

Proof. This follows from the fact that for each pair a, b there is a $c \in A$ such that $Z_c \subset Z_a \cap Z_b$ [4, The proof of Theorem 3.2.13, p. 188]. □

Let (X, τ) be an arbitrary topological space. According to [17], a point $x \in X$ is said to be a θ*-cluster point* of a set $A \subset X$ if and only if $\operatorname{Cl} V \cap A \neq \emptyset$ whenever V is an open neighbourhood of x. Let $|A|_\theta$ denote the set of all θ-cluster points of A; A is said to be θ- *closed* if and only if $|A|_\theta = A$. The above concepts are generally used in the literature (see e.g. [14] and [2]).

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Proposition 3. [3, (2.3)]. *A space X is Hausdorff if and only if for each $p \in X$, $|\{p\}|_\theta = \{p\}$.*

Proposition 4. [3, (2.4)]. *A space is regular if and only if for every $A \subset X$, $|A|_\theta =$ Cl A.*

In the sequel the following theorem frequently will be used.

Theorem 1.1. [6, Theorem 2]. *In any topological space:*

- **(a):** *the empty set and the whole space are θ-closed,*
- **(b):** *arbitrary intersection and finite unions of θ-closed sets are θ-closed,*
- **(c):** *Cl $K \subset |K|_\theta$ for each subset K,*
- **(d):** *a θ-closed subset is closed.*

A subset $A \subset X$ is said to be *θ-open* if $X \diagdown A$ is θ-closed.A subset $A \subset X$ is said to be *regular- open* provided Int (Cl $(A)) = A$.

A Hausdorff space X is *H-closed* [1] if it is closed in any Hausdorff space in which it is embedded.

The following two characterizations are given in [1].

Proposition 5. [1, Theorem 1]. *A Hausdorff space X is H-closed if and only if every family $\{U_\mu : U_\mu$ is open in X, $\mu \in \Omega\}$ with the finite intersection property has the property $\cap\{$ Cl $U_\mu : \mu \in \Omega\} \neq \emptyset$.*

Proposition 6. [1, Theorem 2]. *A Hausdorff space X is H-closed if for each open cover $\{U_\mu : \mu \in M\}$ of X there exists a finite subfamily $\{U_{\mu_1}, ..., U_{\mu_k}\}$ such that $\{$ Cl $U_{\mu_1}, ...,$ Cl $U_{\mu_k}\}$ is a cover of X.*

Proposition 7. [6]. *A Hausdorff space X is H-closed if and only if for every family $\{A_\mu : A_\mu \subset X,\ \mu \in \Omega\ \}$ with the finite intersection property there exists a point $x \in X$ such that Cl $V \cap A \neq \emptyset$ for every open set V containing x and every A_μ.*

The point x is called *θ-accumulation point.* From this characterizations it follows the following lemma frequently used in the paper.

Lemma 1.2. *If X is H-closed, then every family $\{A_\mu,\ \mu \in \Omega\ \}$ of θ-closed subsets of X with the finite intersection property has a non-empty intersection $\cap\{A_\mu,\ \mu \in \Omega\ \}$.*

Proof. Let X be H-closed and let $\{A_\mu,\ \mu \in \Omega\ \}$ be a family of θ-closed subsets of X with the finite intersection property. By Proposition 7 we infer that there exists a θ-accumulation point x such that Cl $V \cap A \neq \emptyset$ for every open set V containing x and every A_μ. This means that $x \in \cap\{A_\mu :,\ \mu \in \Omega\ \}$ since each A_μ is θ-closed. □

Theorem 1.3. [2, (2.4), p.410]. *Disjoint θ-closed subsets of an H-closed space are contained in disjoint open subsets.*

Lemma 1.4. *If $f : X \to Y$ is a continuous mapping, then $f^{-1}(F)$ is θ-closed in X if F is θ-closed in Y.*

Proof. If $x \in X \diagdown f^{-1}(F)$, then $f(x) \notin F$. There exists an open set U such that $f(x) \in U$ and Cl $U \cap F = \emptyset$ since F is θ-closed in Y. The open set $f^{-1}(U)$ contains x and Cl $f^{-1}(U) \cap f^{-1}(F) = \emptyset$ since $f^{-1}($ Cl $U) \cap f^{-1}(F) = \emptyset$. Hence, if $x \in X \diagdown f^{-1}(F)$, then $x \in X \diagdown \left|f^{-1}(F)\right|_\theta$, and, consequently, $f^{-1}(F)$ is θ-closed in X. □

A net $\{x_\mu : \mu \in M\}$ is *eventually* in a set A if and only if there exists a $\mu \in M$ such that $x_\nu \in A$ for each $\nu \geq \mu$ [12, p. 65].

A net $\{x_\mu : \mu \in M\}$ is *frequently* in a set A if and only if for each $\mu \in M$ there is a $\nu \geq \mu$ such that $x_\nu \in A$.

A net in a topological space is said to *θ-converge* (*θ-accumulate*) [6, Definition 3] to a point x in the space if then net is eventually (frequently) in Cl (V) for each open set V about x.

The following two theorems are proved in [17, Lemmas 1, 2, 3]. See also [9].

Theorem 1.5. *A point x in a topological space is in θ-closure of a subset K if and only if there is a net x_a in K which θ-converges to x ($x_a \underset{\theta}{\longrightarrow} x$).*

Theorem 1.6. *A Hausdorff space is H-closed if and only if each net in the space has a θ-convergent subnet.*

In the sequel the following Proposition will be frequently used.

Proposition 8. [3, (2.7), p. 45]. *A θ-closed subset of an H-closed space is H-closed.*

2. Inverse limit of H-closed spaces and mappings with θ-closed graphs

In this Section we consider inverse limit of inverse systems $\mathbf{X} = \{X_a, p_{ab}, A\}$ of H-closed spaces X_a and bonding mappings p_{ab} with θ-closed graphs. Such bonding mappings p_{ab} are special case of multifunction considered in [11].

Let $f : X \to Y$ be a mapping. The graph $G(f)$ of f is

$$G(f) = \{(x, y) \in X \times Y : y = f(x)\}.$$

Theorem 2.1. [11, Theorem 2.3].*The following statements are equivalent for spaces X, Y, and multifunction $\Phi : X \to Y$:*

- **(a):** *The multifunction Φ has a θ-closed graph $G(\Phi)$,*
- **(b):** *For each $(x, y) \in (X \times Y) - G(\Phi)$ there are sets $V \ni x$ in X and $W \ni y$ in Y with $\Phi(\ Cl\ (V)) \cap\ Cl\ (W) = \emptyset$.*

Now we shall prove the following result concerning inverse limit of inverse systems $\mathbf{X} = \{X_a, p_{ab}, A\}$ of H-closed spaces X_a and bonding mappings p_{ab} with θ-closed graph.

Theorem 2.2. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of non-empty H-closed spaces X_a and bonding mappings p_{ab} with θ-closed graphs. Then $X = \lim \mathbf{X}$ is non-empty, θ-closed in $\Pi\{X_a : a \in A\}$ and H-closed.*

Proof. It is known that $\Pi\{X_a : a \in A\}$ is H-closed [4, Problem 3.12.5 (d), p. 283]. Let us prove that $Z_a = \{(x_b) \in \Pi X_a : p_{ab}(x_a) = x_b\}$ is θ-closed for each $a \in A$. To do this we shall prove that $\Pi\{X_a : a \in A\} \setminus Z_a$ is θ -open. Let $y = (y_a) \in \Pi\{X_a : a \in A\} \setminus Z_a$. There exists $b \leq a$ such that $p_{ab}(x_a) \neq x_b$. It follows from Theorem 2.1 that there exists a pair U, V of open sets such that $x_a \in U$, $x_b \in V$ and $p_{ba}(\text{ Cl } U) \cap \text{ Cl } V = \emptyset$ since p_{ba} has a θ-closed graph.

Now $Z = U \times V \times \Pi\{X_c : c \neq a, b\}$ is open set containing y with the property Cl $Z \subset \Pi\{X_a : a \in A\} \setminus Z_a$. This means that $\Pi\{X_a : a \in A\} \setminus Z_a$ θ-open, and, consequently, Z_a is θ-closed. In order to prove that $X = \lim \mathbf{X}$ is non-empty consider the family $\{Z_a : a \in A\}$ of θ-closed sets Z_a. This family has the finite intersection property (Proposition 2). By Lemma 1.2 we infer that $\cap\{Z_a : a \in A\}$ $= \lim \mathbf{X}$ is non-empty. Now, (b) of Theorem 1.1 implies that $\lim \mathbf{X}$ is θ-closed. Finally, from Proposition 8 it follows that $\lim \mathbf{X}$ is H-closed. □

3. Inverse limit of H-closed spaces and strongly continuous bonding mappings

A mapping $f : X \to Y$ is said to be *strongly continuous at* $x \in X$ [15] provided for each neighborhood U of $f(x)$ there is a neighborhood V of x such that $f(\text{ Cl } V) \subset U$. A mapping $f : X \to Y$ is said to be *strongly continuous* provided f is strongly continuous at each point $x \in X$.

If Y is a regular space, then each continuous mapping $f : X \to Y$ is strongly continuous.

Proposition 9. *Let Y be a Hausdorff space. Every strongly continuous mapping $f : X \to Y$ has a θ-closed graph.*

Proof. Let $x \in X$ and $y \in Y$ such that $y \neq f(x)$. There are open disjoint sets U, V in Y such that $y \in U$ and $f(x) \in V$. It is clear that $\text{Cl } U \cap V = \emptyset$. Moreover, there is an open set W containing x such that $p_{ab}(\text{ Cl } W) \subset V$ since f is strongly continuous. Now, for $(x, y) \in (X \times Y) - G\{f\}$ there are sets $W \ni x$ in X and $U \ni y$ in Y with $f(\text{ Cl }(W)) \cap \text{ Cl }(U) = \emptyset$. By Theorem 2.1 the proof is completed. □

Theorem 2.2 and Proposition 9 imply the following result.

Theorem 3.1. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of non-empty H-closed spaces X_a and strongly continuous bonding mappings. Then $X = \lim \mathbf{X}$ is non-empty. Moreover, $X = \lim \mathbf{X}$ is θ-closed in $\Pi\{X_a : a \in A\}$ and H-closed.*

4. Inverse limit of H-closed spaces and θ-closed bonding mappings

In this section we study the inverse systems $\mathbf{X} = \{X_a, p_{ab}, A\}$ with H-closed spaces X_a and θ-closed bonding mappings p_{ab}.

A mapping $f : X \to Y$ is said to be θ-closed if $f(F)$ is θ-closed for each θ-closed subset $F \subset X$.

Remark 4.1. *In* [16, Definition 4.1, p. 490] *is given the following definition. A function f is said to be θ-open if the image of every open set is θ-open. Similarly, a function f is said to be θ -closed if the image of every closed set is θ-closed.*

Lemma 4.2. *Let $f : X \to Y$ be a continuous mapping. The following conditions are equivalent:*

- **(a):** *f is θ-closed,*
- **(b):** *for every $B \subset Y$ and each θ-open set $U \supseteq f^{-1}(B)$ there exists a θ-open set $V \supseteq B$ such that $f^{-1}(V) \subset U$.*

Proof. The proof is similar to the proof of the corresponding theorem for closed mappings [4, p. 52]. □

Now we are ready to prove the following theorem.

Theorem 4.3. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of non-empty H-closed spaces X_a and θ-closed bonding mappings p_{ab}. Then $X = \lim \mathbf{X}$ is non-empty and*

$$p_a(X) = \cap\{p_{ab}(X_b) : b \geq a\}$$

where $p_a : X \to X_a, a \in A$, is a natural projection.

Proof. Let θ_a be a family of all non-empty θ-closed subsets of X_a and let $\mathcal{Y}$ be a family of all collections $Y = \{Y_a : Y_a \in \theta_a, a \in A\}$ such that $p_{ab}(Y_b) \subset Y_a$. The family $\mathcal{Y}$ is non-empty since $\mathbf{X} \in \mathcal{Y}$. For two collections $Y = \{Y_a : Y_a \in \theta_a, a \in A\}$ and $Z = \{Z_a : Z_a \in \theta_a, a \in A\}$ we shall write $Y \geq Z$ if $Y_a \subset Z_a$ for every $a \in A$. It is clear that $(\mathcal{Y}, \geq)$ is a partially ordered set. The remaining part of the proof consists of several steps.

Step 1. *There exists a maximal element in* $(\mathcal{Y}, \geq)$. It suffices to prove that $(\mathcal{Y}, \geq)$ is inductive, i.e., if $L = \{Y^\lambda : \lambda \in \Lambda\}$ is a strictly increasing chain in $(\mathcal{Y}, \geq)$, then there is an element $M \in (\mathcal{Y}, \geq)$ such that $M \geq Y^\lambda$ for every $\lambda \in \Lambda$. We define $M = \{M_a : M_a \in \theta_a, a \in A\}$ such that $M_a = \cap\{Y_a^\lambda : \lambda \in \Lambda\}$. From Lemma 1.2 and Theorem 1.1 it follows that the set M_a is non-empty θ-closed subset of X_a. Moreover, $p_{ab}(M_b) \subset M_a$.

Step 2. *If* $Y = \{Y_a : Y_a \in \theta_a, a \in A\}$ *is a maximal element of* $(\mathcal{Y}, \geq)$, *then* $Y_a = p_{ab}(Y_b)$ *for every pair* $a, b \in A$ *such that* $a \leq b$. Let $Z = \{Z_a : Z_a \in \theta_a, a \in A\}$ be a collection such that $Z_a = \cap\{p_{ab}(Y_b) : b \geq a\}$. Each $p_{ab}(Y_b)$ is θ-closed since p_{ab} is θ-closed and $Y_b \in \theta_b$. By Lemma 1.2 and Theorem 1.1 it follows that the set Z_a is non-empty θ-closed subset of X_a. In order to prove that $Z \in (\mathcal{Y}, \geq)$ it suffices to prove that $p_{ab}(M_b) \subset M_a$. If $a \leq b$ then $p_{ab}(Z_b) \subset \cap\{p_{ab}(p_{bc}(Y_c)) : b \leq c\} = \cap\{p_{ac}(Y_c) : c \geq b\}$. On the other hand, for every $d \geq a$ there is a $c \in A$ such that $c \geq b, d$. It follows that $p_{ac}(Y_c) \subset p_{ad}(Y_d)$. This means that

$$\cap\{p_{ac}(Y_c) : c \geq b\} = \cap\{p_{ad}(Y_d) : c \geq b\} = Z_a.$$

Finally, we have $Z \in (\mathcal{Y}, \geq)$. Moreover, $Z_a \subset Y_a$ for each $a \in A$. This means that $Z = Y$ since Y is maximal.

Step 3. *If* $Y = \{Y_a : Y_a \in \theta_a, a \in A\}$ *is a maximal element of* $(\mathcal{Y}, \geq)$, *then* Y_a *is one-point set for every* $a \in A$. Let $x_a \in Y_a$. Define

$$Z_b = \begin{cases} Y_b \cap p_{ab}^{-1}(x_a) & \text{if} \quad b \geq a, \\ Y_b & \text{if} \quad b \ngeq a. \end{cases}$$

Let us prove that $Z = \{Z_a : Z_a \in \theta_a, a \in A\}$. From Proposition 3 and Lemma 1.4 it follows that $p_{ab}^{-1}(x_a)$ is θ-closed. Then, by Theorem 1.1, we infer that each $Y_b \cap p_{ab}^{-1}(x_a)$ is θ-closed. It is easy to prove that $p_{ab}(Z_b) \subset Z_a$. Hence, $Z \in (\mathcal{Y}, \geq)$. Now, $Z = Y$ since $Z \geq Y$ and Y is maximal. This means $Y_a = \{x_a\}$.

Step 4. $\lim \mathbf{X}$ *is non-empty.* From Step 3 we have $Z = \{Z_a : Z_a \in \theta_a, a \in A\} = \{x_a : a \in A\}$ such that $p_{ab}(x_b) = x_a$ for every pair a, b such that $b \geq a$.

Step 5. *Let us prove* $p_a(X) = \cap\{p_{ab}(X_b) : b \geq a\}$. It is clear that $p_a(X) \subset \cap\{p_{ab}(X_b) : b \geq a\}$. Let us prove that $p_a(X) \supset \cap\{p_{ab}(X_b) : b \geq a\}$. Let $x_a \in \cap\{p_{ab}(X_b) : b \geq a\}$. This means that $Y_b = p_{ab}^{-1}(x_a)$ is non-empty for each $b \geq a$. Moreover, Y_b is θ-closed (Proposition 3 and Lemma 1.4). For each b non-comparable with a, let $Y_b = X_b$. Now, we have a collection $Y = \{Y_a : Y_a \in \theta_a, a \in A\}$ which is evidently in $(\mathcal{Y}, \geq)$. There exists a maximal element $Z = \{Z_a : Z_a \in \theta_a, a \in A\}$ in $(\mathcal{Y}, \geq)$ such that $Z \geq Y$. It follows that each Y_a is some Z_a which is a point $z_a \in X_a$ (Step 3) since Z is maximal. The collections (z_a) is a point of $\lim \mathbf{X}$. Hence, $p_a(X) = \cap\{p_{ab}(X_b) : b \geq a\}$. □

QUESTION 1. Is it true that $X = \lim \mathbf{X}$ in Theorem 4.3 is H-closed?
QUESTION 2. Is every projection $p_a : \lim \mathbf{X} \to \mathbf{X}_a$ θ-closed?
At the end of this section we consider the special kinds of θ-closed mappings.

A mapping $f : X \to Y$ has the *inverse property* provided $f^{-1}(\operatorname{Cl} V) = \operatorname{Cl} f^{-1}(V)$ for every open set $V \subset Y$.

Lemma 4.4. *If $f : X \to Y$ is a closed mapping with the inverse property and if X and Y are H-closed, then f is θ-closed.*

Proof. Let F be a θ-closed subset of X. In order to prove that $f(F)$ is θ-closed we shall prove that $Y \diagdown f(F)$ is θ-open. Let $y \in Y \diagdown f(F)$. Now, $f^{-1}(y)$ is θ-closed subset of X (Lemma 1.4). Using Theorem 1.3 we obtain disjoint open sets U and V such that $F \subset U$ and $f^{-1}(y) \subset V$. It follows that $\operatorname{Cl} V \cap U = \emptyset$. The closeness of f imply the existence of an open set W about y such that $f^{-1}(W) \subset V$. We infer that $\operatorname{Cl} f^{-1}(W) \subset \operatorname{Cl} V$. Moreover, $f^{-1}(\operatorname{Cl} W) \subset \operatorname{Cl} V$. It follows that $f^{-1}(\operatorname{Cl} W) \cap F = \emptyset$, i.e., $\operatorname{Cl} W \cap f(F) = \emptyset$. Hence, if $y \in Y \diagdown f(F)$, then y has a neighborhood W such that $\operatorname{Cl} W \subset Y \diagdown f(F)$, i.e., $Y \diagdown f(F)$ is θ-open and $f(F)$ is θ-closed. □

Each open mapping has the inverse property [4, Exercise 1.4.C., p. 57] . Hence, we have the following corollary.

Corollary 4.5. *If $f : X \to Y$ is a closed and open mapping and if X and Y are H-closed, then f is θ-closed.*

Lemma 4.6. *If X and Y are H-closed, then each strongly continuous mapping $f : X \to Y$ is θ-closed.*

Proof. Let us recall that $f : X \to Y$ is said to be strongly continuous at $x \in X$ [15] provided for each neighborhood U of $f(x)$ there is a neighborhood V of x such that $f(\operatorname{Cl} V) \subset U$. A mapping $f : X \to Y$ is said to be strongly continuous provided f is strongly continuous at each point $x \in X$. Now, let us prove Lemma.

Let F be a θ-closed subset of X. We have to prove that $f(F)$ is a θ-closed subset of Y. Suppose that it is not θ-closed. There is a point $y \in |\, f(F)|_\theta \diagdown f(F)$. By Theorem 1.5 we infer that there is a net $\{y_a : y_a \in f(F), a \in A\}$ which θ-converges to y. Now there is a net $\{x_a : x_a \in F, f(x_a) = y_a\}$. By Theorem 1.6 we may assume that this net is θ-convergent to some point $x \in X$. From Theorem 1.5 it follows that $x \in F$ since F is θ-closed. It is clear that $f(x)$ is θ-limit of $\{f(x_a) : x_a \in F\} = \{y_a : y_a \in f(F), a \in A\}$. We infer that $f(x) = y$ since, in the opposite case, $f(x)$ and y have disjoint neighborhoods U and V such that $f(x) \in U$ and there is a neighborhood W such that $f(\operatorname{Cl} W) \subset U$. This means that a net $\{y_a : y_a \in f(F), a \in A\}$ is not eventually in $\operatorname{Cl} V$. This is impossible. Hence, $f(x) = y$ From $x \in F$ it follows that $f(x) \in f(F)$. Hence $y \in f(F)$ and $f(F)$ is θ-closed. The proof is completed. □

Lemma 4.7. *If Y is Urysohn and X H-closed, then each continuous mapping $f : X \to Y$ is θ-closed.*

Proof. Let F be a θ-closed subset of X. We have to prove that $f(F)$ is a θ-closed subset of Y. Suppose that it is not θ-closed. There is a point $y \in |f(F)|_\theta \diagdown f(F)$. By Theorem 1.5 we infer that there is a net $\{y_a : y_a \in f(F), a \in A\}$ which θ-converges to y. Now there is a net $\{x_a : x_a \in F, f(x_a) = y_a\}$. By Theorem 1.6 we may assume that this net is θ-convergent to some point $x \in X$. From Theorem 1.5 it follows that $x \in F$ since F is θ-closed. It is clear that $f(x)$ is θ-limit of $\{f(x_a) : x_a \in F\}$ $= \{y_a : y_a \in f(F), a \in A\}$. We infer that $f(x) = y$ since in Urysohn space a net has only one θ-limit. From $x \in F$ it follows that $f(x) \in f(F)$. Hence $y \in f(F)$ and $f(F)$ is θ-closed. The proof is completed □

A function $f : X \to Y$ is *almost closed* [2] if for any set $A \subset X$ we have $f(|A|_\theta) = |f(A)|_\theta$.

Now we shall prove the following theorem.

Theorem 4.8. *Each almost closed function is θ-closed.*

Proof. If A is θ-closed, then $A = |A|_\theta$. Now we have $f(|A|_\theta) = |f(A)|_\theta$ or $f(A) = |f(A)|_\theta$. This means that $f(A)$ is θ-closed. Hence f is θ-closed. □

Corollary 4.9. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of non-empty H-closed spaces X_a and closed bonding mappings p_{ab} with the inverse property. Then $X = \lim \mathbf{X}$ is non-empty and H-closed.*

Proof. Lemma 4.4 and Theorem 4.3 imply the Corollary. H-closenes of $\lim \mathbf{X}$ it follows from Theorems 3.3 and 3.7 of [5]. □

5. Inverse systems of nearly-compact spaces

We say that a space X is an *Urysohn space* ([7], [10]) if for every pair $x, y, x \neq y$, of points of X there exist open sets V and W about x and y such that $\text{Cl } V \cap \text{Cl } W = \emptyset$.

A Hausdorff space is *nearly-compact* [8] if every open cover if every open cover $\{U_\mu : \mu \in M\}$ has a finite subcollection $\{U_{\mu_1}, ..., U_{\mu_n}\}$ such that $\text{Int Cl } U_{\mu_1} \cup ... \cup \text{Int Cl } U_{\mu_n} = X$. Every nearly-compact space is H-closed.

Lemma 5.1. [8]. *A space X is nearly-compact if and only if it is H-closed and Urysohn.*

If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an inverse system of nearly-compact spaces, then θ-closeness of bonding mappings p_{ab} in Theorem 4.3 follows from Lemma 4.7, but we shall give the alternate proof of the following theorem.

Theorem 5.2. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of non-empty nearly-compact spaces X_a. Then $X = \lim \mathbf{X}$ is non-empty, θ-closed in $\Pi\{X_a : a \in A\}$ and nearly-compact.*

Proof. Let us observe that $\Pi\{X_a : a \in A\}$ is H-closed [4, Problem 3.12.5 (d), p. 283]. Let us prove that $Y_a = \{(x_b) \in \Pi X_a : p_{ab}(x_a) = x_b\}$ θ-closed for each $a \in A$. To do this we shall prove that $\Pi X_a \setminus Y_a$ θ-open. Let $y = (y_a) \in \Pi X_a \setminus Y_a$. There exists $b \leq a$ such that $p_{ab}(x_a) \neq x_b$. It follows that there exists a pair U, V of open sets such that $x_b \in U$, $p_{ab}(x_a) \in V$ and $\text{Cl } U \cap \text{Cl } V = \emptyset$ since X_b is Urysohn. Moreover, there is an open set W containing x_a such that $p_{ab}(\text{Cl } W) \subset \text{Cl } V$. Now $Z = U \times W \times \Pi\{X_c : c \neq a, b\}$ is open set containing y with the property $\text{Cl } Z \subset \Pi X_a \setminus Y_a$. This means that To $\Pi X_a \setminus Y_a$ θ-open, and, consequently, Y_a is θ-closed. In order to prove that $X = \lim \mathbf{X}$ is non-empty consider the family $\{Y_a : a \in A\}$ of θ-closed sets Y_a. This family has the finite intersection property (Proposition 2). By Lemma 1.2 we infer that $\cap\{Y_a : a \in A\} = \lim \mathbf{X}$ is non-empty. It is θ-closed by Theorem 1.1 and H-closed by Proposition 8. Moreover, $\lim \mathbf{X}$ is Urysohn and, consequently, nearly-compact. □

6. Inverse systems with semi-open bonding mappings

A mapping $f : X \to Y$ is said to be *semi-open* provided $\text{Int}\, f(U) \neq \emptyset$ for each non-empty open $U \subset X$.

Theorem 6.1. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of non-empty H-closed spaces X_a and semi-open bonding mappings. Then $X = \lim \mathbf{X}$ is non-empty and H-closed.*

Proof. The proof is broken into several steps.

Step 1. By virtue of [13, Theorem 2, p. 10] we can assume that A is cofinite, i.e., for each $a \in A$ the set of all predecessors of a is finite set.

Step 2. *The sets*

$$Z_a = \{\{x_a\} \in \Pi X_a : p_{ab}(x_b) = x_a,\ a \leq b\}$$

have non-empty interior. Let $a_1, ..., a_k$ be a set of all predecessors of a. If $U \subset X_a$ is open set, then $\text{Int}\, p_{a_1 a}(U) \times ... \times \text{Int}\, p_{a_k a}(U) \times U \times \Pi\{X_b : b \notin \{a_1, ..., a_k, a\}\}$ is an open set contained in Z_a. Hence, $\text{Int}\, Z_a$ is non-empty for each $a \in A$.

Step 3. *The family $\{\text{Int}\, Z_a : a \in A\}$ has the finite intersection property.* This follows from the fact that for each pair a, b there is a $c \in A$ such that $Z_c \subset Z_a \cap Z_b$ and, consequently, $\text{Int}\, Z_c \subset \text{Int}\, Z_a \cap \text{Int}\, Z_b$.

Step 4. $\cap\{\text{Cl}\ \text{Int}\, Z_a : a \in A\}$ is non-empty. This follows from Proposition 5.

Step 5. Now $\lim \mathbf{X} = \cap \{Z_a : a \in A\} \supset \cap\{\text{Cl}\ \text{Int}\, Z_a : a \in A\}$. This means that $\lim \mathbf{X}$ is non-empty and the proof of non-emptiness is completed.

Step 6. $X = \lim \mathbf{X}$ *is H-closed.* Let $\mathcal{U} = \{U_\mu : \mu \in M\}$ be a maximal family of open sets of X with the finite intersection property. From the definition of topology on X it follows that there is an $a(\mu) \in A$ such that $\text{Int}\, f_{a(\mu)}(U_\mu)$ is non-empty. By virtue of the semi-openness of p_{ab} we infer that $\text{Int}\, f_a(U_\mu) \neq \emptyset$ for every $a \in A$ and every $\mu \in M$. This means that a family $\{\text{Int}\, f_a(U_\mu) : \mu \in M\}$ is a family with the finite intersection property. Let us prove that this family is maximal. If U is an open set which intersects every set $\text{Int}\, f_a(U_\mu), \mu \in M$, then $p_a^{-1}(U) \in \mathcal{U}$ since $p_a^{-1}(U)$ intersects every U_μ. This means that $U \in \{\text{Int}\, f_a(U_\mu) : \mu \in M\}$. Hence, $\{\text{Int}\, f_a(U_\mu) : \mu \in M\}$ is maximal. From the H-closeness of X_a and Proposition 5 it follows that there is a point $x_a = \cap\{\text{Cl}\ \text{Int}\, f_a(U_\mu) : \mu \in M\}$. It is obvious that $p_{ab}(x_b) = x_a$ for every $b \geq a$.Now, $x = (x_a : a \in A)$ is a point of $\lim \mathbf{X}$ and $x \in \cap\{\text{Cl}\, U_\mu) : \mu \in M\}$. By Proposition 5 $\lim \mathbf{X}$ is H-closed and the proof is completed. □

We close this Section with some corollaries of Theorem 6.1.

Corollary 6.2. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of non-empty H-closed spaces X_a and open bonding mappings. Then $X = \lim \mathbf{X}$ is non-empty and H-closed.*

Remark 6.3. *For another proof of this corollary see* [18].

A mapping $f : X \to Y$ is an *irreducible mapping* if the set $f^{\#}(U) = \{y \in Y : f^{-1}(y) \subset U\}$ is non-empty for every non-empty open se $U \subset X$. If $f : X \to Y$ is a closed and irreducible mapping, then $f^{\#}(U)$ is open and non-empty. Hence, a closed and irreducible mapping is semi-open. Theorem 6.1 now gives the following corollary.

Corollary 6.4. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of non-empty H-closed spaces X_a and closed irreducible bonding mappings. Then $X = \lim \mathbf{X}$ is non-empty and H-closed.*

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References

[1] P. S. Aleksandroff et P. S. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verh. Akademie Amsterdam, Deel XIV, Nr. 1 (1929), 1-96.
[2] R. F. Dickman, Jr. and J. R. Porter, *θ -closed subsets of Hausdorff spaces*, Pac. J. Math. 59 (1975), 407-415.
[3] R. F. Dickman, Jr. and J. R. Porter, *θ -perfect and θ-absolutely closed function,* Illinois J. Math 21 (1977), 42-60.
[4] R. Engelking, *General Topology*, PWN,Warszawa, 1977.
[5] L. M. Friedler and D. H. Pettey, *Inverse limits and mappings of minimal topological spaces*, Pac. J. Math. 71 (1977), 429-448.
[6] L. L. Herrington and P. E. Long, *Characterizations of H-closed spaces*, Proc. Amer. Math. Soc. 48 (1975), 469-475.
[7] L. L. Herrington, *Characterizations of Urysohn-closed spaces*, Proc. Amer. Math. Soc. 55 (1976), 435-439.
[8] L. L. Herrington, *Properties of nearly-compact spaces*, Proc. Amer. Math. Soc. 45 (1974), 431-436.
[9] J. E. Joseph, *On H-closed spaces*, Proc. Amer. Math. Soc. 55 (1976), 223-226.
[10] J. E. Joseph, *On Urysohn-closed and minimal Urysohn spaces*, Proc. Amer. Math. Soc. 68 (1978), 235-242.
[11] J. E. Joseph, *Multifunctions and graphs*, Pacific J. Math, Vol. 79, 1978, 509-529.
[12] J. L. Kelley, General topology, D. van Nostrand 1963.
[13] S. Mardesic and J. Segal, Shape theory. The inverse system approach, North-Holland Publishing Company, (1982).
[14] T. Noiri, *Strongly θ-precontinuous functions*, Acta Math. Hungar., 90 (2001), 307-316.
[15] M. Saleh, *On θ-Continuity And Strong θ-Continuity*, Applied Mathematics E-Notes 3 (2003), 42-48.
[16] M. Saleh, *On θ-closed sets and some forms of continuity*, Archivum Mathematicum (Brno), 40 (2004), 383-393.
[17] N. V. Veličko, *H-closed topological spaces*, Mat. Sb., 70 (112) (1966), 98-112.
[18] T. O. Vinson, Jr. and R. F. Dickman, Jr., *Inverse limits and absolutes of H-closed spaces*, Proc. Amer. Math. Soc. 66 (1977), 351-358.

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