

ON THE INSTABILITY OF SOLUTIONS TO A CERTAIN CLASS
 OF NON-AUTONOMOUS AND NON-LINEAR ORDINARY
 VECTOR DIFFERENTIAL EQUATIONS OF SIXTH ORDER

CEMIL TUNÇ

ABSTRACT. The aim of the present paper is to establish a new result, which guarantees the instability of zero solution to a certain class of non-autonomous ordinary differential equations of sixth order. Our result improves some known results in the literature for non- autonomous case (see, [20, Theorem 4], [31, Theorem 1]).

1. INTRODUCTION

Consider the non-autonomous and non-linear vector differential equation of sixth order:

$$(1) \quad \begin{aligned} X^{(6)} + AX^{(5)} + BX^{(4)} + C\ddot{X} + \Phi(t, X, \dot{X}, \ddot{X}, \ddot{X}, X^{(4)}, X^{(5)})\ddot{X} \\ + \Psi(X)\dot{X} + H(t, X, \dot{X}, \ddot{X}, \ddot{X}, X^{(4)}, X^{(5)})X = 0, \end{aligned}$$

in which $t \in \mathfrak{R}_+$, $\mathfrak{R}_+ = [0, \infty)$ and $X \in \mathfrak{R}^n$; A , B and C are constant $n \times n$ -real symmetric matrices; Φ , Ψ and H are continuous $n \times n$ -symmetric real matrix functions depending, in each case, on the arguments shown in (1). Let $J(\Psi(X)X | X)$ denote the linear operator from the matrix function $\Psi(X)$ to the matrix

$$J(\Psi(X)X | X) = \left(\frac{\partial}{\partial x_j} \sum_{k=1}^n \psi_{ik} x_k \right) = \Psi(X) + \left(\sum_{k=1}^n \frac{\partial \psi_{ik}}{\partial x_j} x_k \right),$$

where (x_1, x_2, \dots, x_n) and (ψ_{ik}) are components of X and Ψ , respectively. It is assumed that the matrix $J(\Psi(X)X | X)$ exists and is symmetric and continuous. From the relevant literature, it can be followed that, so far, many problems about the instability of solutions of various scalar and vector linear or nonlinear differential equations of third-, fourth-, fifth-, sixth-, seventh and eighth order have been investigated by researchers. For some papers carried out on the topic, one can refer to the book of Reissig et al [15] and the papers of Bereketoğlu [2], Ezeilo ([3], [4], [5], [6], [7]), Liao and Lu [9], Li and Yu [10], Li and Duan [11], Lu and Liao [12], Lu [13], Sadek ([16], [17]), Skrapek ([18], [19]), Tejumola [20], Tiryaki ([21], [22], [23]), Tunç ([24], [25], [26], [27], [28], [29], [30], [31], [32]), C.Tunç and E. Tunç ([33], [34], [35], [36]), C. Tunç and H. Seveli [37], E. Tunç [38] and the references listed therein. Throughout all the papers mentioned above the Lyapunov's second

2000 *Mathematics Subject Classification.* 34D05; 34D20.

Key words and phrases. Non-autonomous differential equation, sixth order, instability.

(or direct) method [14] is used as a basic tool to prove the result established there, and it will be also used to verify our result, which will be given hereafter. The motivation for the present work has been inspired especially by the papers in [20], [31] and the papers mentioned above. It should be noted that in [20], Tejumola investigated the instability of the trivial solution of the following sixth order scalar nonlinear differential equation of the type

$$x^{(6)} + a_1x^{(5)} + a_2x^{(4)} + a_3\ddot{x} + \varphi_4(x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)}, x^{(5)})\ddot{x} \\ + \varphi_5(x)\dot{x} + \varphi_6(x, \dot{x}, \ddot{x}, \ddot{x}, x^{(4)}, x^{(5)}) = 0.$$

He proved a result on the subject. Recently, in [31], Tunç investigated the instability of the trivial solution of sixth order nonlinear vector differential equation of the form

$$X^{(6)} + AX^{(5)} + BX^{(4)} + C\ddot{X} + \Phi(X, \dot{X}, \ddot{X}, \ddot{X}, X^{(4)}, X^{(5)})\ddot{X} \\ + \Psi(X)\dot{X} + H(X, \dot{X}, \ddot{X}, \ddot{X}, X^{(4)}, X^{(5)})X = 0.$$

Clearly, (1) is a non-autonomous differential equation, that is, (1) is different from the above equations and ones considered in the literature, see also the papers mentioned above.

Throughout this paper, the symbol $\langle X, Y \rangle$ is used to denote the usual scalar product in \mathfrak{R}^n , that is, $\langle X, Y \rangle = \sum_{i=1}^n x_i y_i$, thus $\langle X, X \rangle = \|X\|^2$, and $\lambda_i(A)$, ($i = 1, 2, \dots, n$), are the eigenvalues of the $n \times n$ - matrix A .

We take into consideration, in place of (1), the equivalent differential system

$$\dot{X} = Y, \dot{Y} = Z, \dot{Z} = S, \dot{S} = T, \dot{T} = U, \\ (2) \quad \dot{U} = -AU - BT - CS - \Phi(t, X, Y, Z, S, T, U)Z \\ - \Psi(X)Y - H(t, X, Y, Z, S, T, U)X,$$

which was obtained as usual by setting $\dot{X} = Y$, $\ddot{X} = Z$, $\ddot{\ddot{X}} = S$, $X^{(4)} = T$, $X^{(5)} = U$ by (1).

2. PRELIMINARIES

In order to reach our main result, we will give a basic theorem for the general non-autonomous differential system and two well-known lemmas which play an essential role in the proof of our main result. Consider the differential system

$$(3) \quad \dot{x} = f(t, x), x(t_0) = x_0, t \geq 0,$$

where $f \in C[R_+ \times S_{(\rho)}, \mathfrak{R}^n]$ and $S_{(\rho)} = [x \in \mathfrak{R}^n : |x| < \rho]$. Assume, for convenience that a solution $x(t) = x(t, t_0, x_0)$ of (3) exists and is unique for $t \geq t_0$ and $f(t, 0) = 0$ so that we have trivial solution $x = 0$. Let K denote a class of the functions as $K = [\sigma \in C[[t_0, \rho), \mathfrak{R}_+]]$ such that $\sigma(t)$ is strictly increasing and $\sigma(0) = 0$.

Now, we state the following fundamental instability theorem.

Theorem 1. Assume that there exists a $t_0 \in \mathfrak{R}_+$ and an open set $U \subset S_{(\rho)}$ such that

$$V \in C^1[[t_0, \infty) \times S_{(\rho)}, \mathfrak{R}_+] \text{ for } (t, x) \text{ from } [t_0, \infty) \times U,$$

- (i) $0 < V(t, x) \leq a(|x|)$, $a \in K$;
- (ii) either $V'(t, x) \geq b(|x|)$, $b \in K$, $K = [\sigma \in C[[t_0, \rho), \mathfrak{R}_+]]$ such that $\sigma(t)$ is strictly increasing and $\sigma(0) = 0$ or $V'(t, x) = CV(t, x) + \omega(t, x)$, where $C > 0$ and $\omega \in C[[t_0, \infty) \times U, \mathfrak{R}_+]$;
- (iii) $V(t, x) = 0$ on $[t_0, \infty) \times (\partial U \cap S_{(\rho)})$, ∂U denotes boundary of U and $0 \in \partial U$.

Then the trivial solution $x = 0$ of system (3) is unstable.

Proof. See Lakshmikantham et al. [Theorem 1.1.9, 8].

Lemma 1. Let A be a real symmetric $n \times n$ -matrix and $a' \geq \lambda_i(A) \geq a > 0$ ($i = 1, 2, \dots, n$) , where a' , a are constants. Then

$$a' \langle X, X \rangle \geq \langle AX, X \rangle \geq a \langle X, X \rangle$$

and

$$a'^2 \langle X, X \rangle \geq \langle AX, AX \rangle \geq a^2 \langle X, X \rangle$$

Proof. See Bellman[1].

Lemma 2. Let Q , D be any two real $n \times n$ commuting symmetric matrices. Then

(i) the eigenvalues $\lambda_i(QD)$, ($i = 1, 2, \dots, n$) , of the product matrix QD are real and satisfy

$$\max_{1 \leq j, k \leq n} \lambda_j(Q)\lambda_k(D) \geq \lambda_i(QD) \geq \min_{1 \leq j, k \leq n} \lambda_j(Q)\lambda_k(D) ;$$

(ii) the eigenvalues $\lambda_i(Q + D)$, ($i = 1, 2, \dots, n$) , of the sum of matrices Q and D are real and satisfy

$$\left\{ \max_{1 \leq j \leq n} \lambda_j(Q) + \max_{1 \leq k \leq n} \lambda_k(D) \right\} \geq \lambda_i(Q + D) \geq \left\{ \min_{1 \leq j \leq n} \lambda_j(Q) + \min_{1 \leq k \leq n} \lambda_k(D) \right\} ,$$

where $\lambda_j(Q)$ and $\lambda_k(D)$ are, respectively, the eigenvalues of matrices Q and D .

Proof. See Bellman[1].

3. MAIN RESULT

We establish the following theorem:

Theorem 2. In addition to the basic assumptions imposed on A , B , C , Φ , Ψ and H that appeared in (2), we assume that the following conditions hold: There are constants a_1 , a_2 and a_5 such that

$$\lambda_i(A) \geq a_1 > 0 , \lambda_i(B) \leq a_2 < 0 , |\lambda_i(\Psi(X))| \leq a_5 , (a_5 > 0) ,$$

and

$$\lambda_i(H(t, X, Y, Z, S, T, U)) < \frac{1}{4a_2} [\lambda_i(\Phi(t, X, Y, Z, S, T, U))]^2 , (i = 1, 2, \dots, n) ,$$

for all $t \in \mathfrak{R}_+$ and X , Y , Z , S , T , $U \in \mathfrak{R}^n$.

Then the zero solution of the system (2) is unstable.

Remark. It should be noted that there is no restriction on the eigenvalues of the matrix C in the system (2).

Proof. To prove Theorem 2, we construct a scalar differentiable Lyapunov function $V_0 = V_0(t, X, Y, Z, S, T, U)$. This function, V_0 , is defined as follows:

$$\begin{aligned} V_0 = & \langle X, U \rangle + \langle X, AT \rangle + \langle X, BS \rangle + \langle X, CZ \rangle \\ & - \langle Y, T \rangle - \langle Y, AS \rangle - \langle Y, BZ \rangle + \langle Z, S \rangle - \frac{1}{2} \langle Y, CY \rangle \\ & + \frac{1}{2} \langle Z, AZ \rangle + \int_0^1 \langle \Psi(\sigma X) X, X \rangle d\sigma. \end{aligned}$$

Clearly, $V_0(t, 0, 0, 0, 0, 0, 0) = 0$ on $[t_0, \infty)$. Now, subject to the assumptions of Theorem 2, it is a straightforward calculation to see that

$$\begin{aligned} V_0(t, 0, 0, \varepsilon, \varepsilon, 0, 0) &= \frac{1}{2} \langle \varepsilon, A\varepsilon \rangle + \langle \varepsilon, \varepsilon \rangle \\ &\geq \frac{1}{2} \langle \varepsilon, a_1 \varepsilon \rangle + \langle \varepsilon, \varepsilon \rangle \\ &= \left(\frac{1}{2}a_1 + 1\right) \|\varepsilon\|^2 > 0 \end{aligned}$$

for all arbitrary, $\varepsilon \neq 0$, $\varepsilon \in \mathfrak{R}^n$. In view of the function $V_0 = V_0(t, X, Y, Z, S, T, U)$, the assumptions of Theorem 2, the properties of symmetric matrices, Lemma 1, Lemma 2 and Cauchy-Schwarz inequality $|\langle X, Y \rangle| \leq \|X\| \|Y\|$, one can easily obtain that there is a positive constant K_1 such that

$$V_0(t, X, Y, Z, S, T, U) \leq K_1 \left(\|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|S\|^2 + \|T\|^2 + \|U\|^2 \right).$$

These show that assumption (i) of Theorem 1 holds.

Now, let $(X, Y, Z, S, T, U) = (X(t), Y(t), Z(t), S(t), T(t), U(t))$ be an arbitrary solution of system (2). By an elementary differentiation along the solution paths of the system (2), it can be verified that

$$\begin{aligned} \dot{V}_0 = \frac{d}{dt} V_0(t, X, Y, Z, S, T, U) &= - \langle \Phi(t, X, Y, Z, S, T, U) Z, X \rangle \\ &\quad - \langle H(t, X, Y, Z, S, T, U) X, X \rangle \\ (4) \quad &\quad - \langle BZ, Z \rangle + \langle S, S \rangle - \langle \Psi(X) Y, X \rangle \\ &\quad + \frac{d}{dt} \int_0^1 \langle \Psi(\sigma X) X, X \rangle d\sigma. \end{aligned}$$

Check that

$$\begin{aligned} (5) \quad \frac{d}{dt} \int_0^1 \langle \Psi(\sigma X) X, X \rangle d\sigma &= \int_0^1 \langle \Psi(\sigma X) X, Y \rangle d\sigma + \int_0^1 \langle \sigma J(\Psi(\sigma X) X | \sigma X) Y, X \rangle d\sigma \\ &= \int_0^1 \langle \Psi(\sigma X) X, Y \rangle d\sigma + \int_0^1 \sigma \langle J(\Psi(\sigma X) X | \sigma X) X, Y \rangle d\sigma \\ &= \int_0^1 \langle \Psi(\sigma X) X, Y \rangle d\sigma + \int_0^1 \sigma \frac{\partial}{\partial \sigma} \langle \Psi(\sigma X) X, Y \rangle d\sigma \\ &= \sigma \langle \Psi(\sigma X) X, Y \rangle \Big|_0^1 = \langle \Psi(X) X, Y \rangle. \end{aligned}$$

Combining the estimate (5) with (4), we deduce that

$$\begin{aligned}\dot{V}_0 = & -\langle \Phi(t, X, Y, Z, S, T, U)Z, X \rangle - \langle H(t, X, Y, Z, S, T, U)X, X \rangle \\ & - \langle BZ, Z \rangle + \langle S, S \rangle\end{aligned}$$

Hence, the assumptions of Theorem 2 and the fact $\langle S, S \rangle = \|S\|^2$ imply that

$$\begin{aligned}\dot{V}_0 & \geq -\langle \Phi(t, X, Y, Z, S, T, U)Z, X \rangle - \langle H(t, X, Y, Z, S, T, U)X, X \rangle - a_2 \langle Z, Z \rangle \\ & = -a_2 \left\| Z + \frac{1}{2a_2} \Phi(t, X, Y, Z, S, T, U)X \right\|^2 - \langle H(t, X, Y, Z, S, T, U)X, X \rangle \\ & \quad + \frac{1}{4a_2} \langle \Phi(t, X, Y, Z, S, T, U)X, \Phi(t, X, Y, Z, S, T, U)X \rangle \\ & \geq -\langle H(t, X, Y, Z, S, T, U)X, X \rangle \\ & \quad + \frac{1}{4a_2} \langle \Phi(t, X, Y, Z, S, T, U)X, \Phi(t, X, Y, Z, S, T, U)X \rangle > 0.\end{aligned}$$

Thus, the assumptions of the theorem imply that $\dot{V}_0(t) \geq K_2 \|X\|^2$ for all $t \geq 0$, where K_2 is a positive constant, say infinite inferior limit of the function \dot{V}_0 . Besides, $\dot{V}_0 = 0$ ($t \geq 0$) necessarily implies that $X = 0$ for all $t \geq 0$, and therefore also that $Z = \dot{Y} = 0$, $S = \dot{Y} = 0$, $T = \ddot{Y} = 0$, $U = Y^{(4)} = 0$ for all $t \geq 0$. Hence

$$X = Y = Z = S = T = U = 0 \text{ for all } t \geq 0.$$

Therefore, subject to the assumptions of the theorem the function V_0 has the entire the criteria of Theorem 1, Lakshmikantham et al. [Theorem 1.1.9, 8]. Thus, the basic properties of the function $V_0(t, X, Y, Z, S, T, U)$, which are proved just above verify that the zero solution of the system (2) is unstable, see also Lakshmikantham et al. [Theorem 1.1.9, 8]. The system of equations (2) is equivalent to differential equation (1) and the proof of Theorem 2 is now complete.

REFERENCES

- [1] Bellman, R., "Introduction to matrix analysis". Reprint of the second (1970) edition. With a foreword by Gene Golub. Classics in Applied Mathematics, 19. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1997.
- [2] Bereketoglu, H., "On the instability of trivial solutions of a class of eighth-order differential equations", *Indian J. Pure. Appl. Math.*, no.3, **22** (1991) 199-202.
- [3] Ezeilo, J. O.C., "An instability theorem for a certain fourth order differential equation", *Bull. London Math. Soc.*, no. 2, **10**(1978) 184-185.
- [4] Ezeilo, J. O.C., "Instability theorems for certain fifth-order differential equations", *Math. Proc. Cambridge Philos. Soc.*, no. 2, **84**(1978) 343-350.
- [5] Ezeilo, J. O.C., "A further instability theorem for a certain fifth-order differential equation", *Math. Proc. Cambridge Philos. Soc.*, no. 3, **86**(1979) 491-493.
- [6] Ezeilo, J. O.C., "Extension of certain instability theorems for some fourth and fifth order differential equations", *Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur.*, no. 4, (**8**) **66**(1979) 239-242.
- [7] Ezeilo, J. O.C., "An instability theorem for a certain sixth order differential equation", *J. Austral. Math. Soc. Ser. A*, no.1, **32**(1982) 129-133.
- [8] Lakshmikantham, V., Matrosov, V.M., Sivasundaram, S., Vector Lyapunov functions and stability analysis of nonlinear systems. Mathematics and its Applications, 63. Kluwer Academic Publishers Group, Dordrecht, 1991.

- [9] Liao, Z.H. and Lu,D., "Instability of solution for the third order linear differential equation with varied coefficient", *Appl. Math. Mech.* (English Ed.), no. 10, **9** (1988) 969-984; translated from *Appl. Math. Mech.*, no. 10, 9 (1988) 909-923 (Chinese).
- [10] Li, W.J. and Yu, Y.H., "Instability theorems for some fourth-order and fifth-order differential equations", (Chinese) *J. Xinjiang Univ. Natur. Sci.*, no. 2, **7**(1990) 7-10.
- [11] Li,W. J. and Duan, K.C., "Instability theorems for some nonlinear differential systems of fifth order", *J. Xinjiang Univ. Natur. Sci.*, no. 3, **17**(2000) 1-5.
- [12] Lu, D. and Liao,Z.H., "Instability of solution for the fourth order linear differential equation with varied coefficient", *Appl. Math. Mech.* (English Ed.), no. 5, **14** (1993) 481-497; translated from *Appl. Math. Mech.*, no. 5, 14 (1993) 455-469 (Chinese).
- [13] Lu, D. "Instability of solution for a class of the third order nonlinear differential equation", *Appl. Math. Mech.* (English Ed.), no. 12, **16** (1995) 1185-1200; translated from *Appl. Math. Mech.*, no. 12, 16 (1995) 1101-1114 (Chinese).
- [14] Lyapunov, A.M., *Stability of Motion*, Academic Press, London, 1966.
- [15] Reissig,R., Sansone, G. and Conti, R. *Non-linear Differential Equations of Higher Order*. Translated from the German, Noordhoff International Publishing, Leyden, 1974.
- [16] Sadek, A.I. "An instability theorem for a certain seventh-order differential equation", *Ann. Differential Equations*, no. 1, **19** (2003) 1-5.
- [17] Sadek, A.I. "Instability results for certain systems of fourth and fifth order differential equations", *Appl. Math. Comput.*, no. 2-3, **145** (2003) 541-549.
- [18] Skrapek, W.A. "Instability results for fourth-order differential equations", *Proc. Roy. Soc. Edinburgh Sect. A* , no. 3-4, **85**(1980) 247-250.
- [19] Skrapek, W.A. "Some instability theorems for third order ordinary differential equations", *Math. Nachr.*, **96**(1980) 113-117.
- [20] Tejumola, H.O. "Instability and periodic solutions of certain nonlinear differential equations of orders six and seven", *Ordinary Differential Equations (Abuja, 2000)*, 56-65, *Proc. Natl. Math. Cent. Abuja Niger.*, 1.1, *Natl. Math. Cent.*, Abuja, 2000.
- [21] Tiryaki, A. "Extension of an instability theorem for a certain fourth order differential equation", *Bull. Inst. Math. Acad. Sinica.*, no.2, **16** (1988) 163-165.
- [22] Tiryaki, A. "An instability theorem for a certain sixth order differential equation", *Indian J. Pure. Appl. Math.*, no.4, **21**(1990) 330-333.
- [23] Tiryaki, A. "Extension of an instability theorem for a certain fifth order differential equation", *National Mathematics Symposium (Trabzon, 1987)*, J. Karadeniz Tech. Univ. Fac. Arts Sci. Ser. Math. Phys., 11, (1988, 1989) 225-227.
- [24] Tunc, C. "An instability theorem for a certain vector differential equation of the fourth order", *JIPAM. J. Inequal. Pure Appl. Math.* **5** (2004), no. 1, Article 5, 5 pp.
- [25] Tunc, C. "An instability result for certain system of sixth order differential equations", *Appl. Math. Comput.*, no.2, **157** (2004) 477-481.
- [26] Tunc, C. "On the instability of solutions of certain nonlinear vector differential equations of fifth order", *Panamer. Math. J.*, **14** (2004), no. 4, 25-30.
- [27] Tunc, C. "On the instability of certain sixth-order nonlinear differential equations", *Electron. J. Diff. Eqns.*, Vol. **2004** , no. 117, (2004) 1-6.
- [28] Tunc, C. "A further instability result for a certain vector differential equation of fourth order", *Int. J. Math. Game Theory Algebra* **15** (2006), no. 5, 489-495.
- [29] Tunc, C. "An instability result for a certain non-autonomous vector differential equation of fifth-order", *Panamer. Math. J.* 15 (2005), no.3, 51-58.
- [30] Tunc, C. "Instability of solutions of a certain non-autonomous vector differential equation of eighth-order", *Ann. Differential Equations* **22** (2006), no. 1, 7-12.
- [31] Tunc, C. "New results about instability of nonlinear ordinary vector differential equations of sixth and seventh orders", *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.***14**(2007), no.1, 123-136.
- [32] Tunc, C. "Some instability results on certain third order nonlinear vector differential equations", *Bull. Inst. Math. Acad. Sin. (N.S.)* **2** (2007), no. 1, 109-122.
- [33] Tunc, C. and Tunc, E. "A result on the instability of solutions of certain non-autonomous vector differential equations of fourth order", *East-West J. Math.* **6** (2004), no. 2, 153-160.
- [34] Tunc, C. and Tunc, E., "Instability of solutions of certain nonlinear vector differential equations of order seven", *Iran. J. Sci. Technol. Trans. A Sci.* **29** (2005), no. 3, 1-7.

- [35] Tunc, C. and Tunc, E., "An instability theorem for a class of eighth-order differential equations". (Russian) *Differ. Uravn.* **42** (2006), no. 1, 135-138, 143; translation in *Differ. Equ.* **42** (2006), no. 1, 150-154.
- [36] Tunc, C. and Tunc, E., "Instability results for certain third order nonlinear vector differential equations", *Electron. J. Differential Equations* 2006, no. 109, 10 pp.
- [37] Tunc, C. and Sevlı, H. "On the instability of solutions of certain fifth order nonlinear differential equations", *Mem. Differential Equations Math. Phys.* **35** (2005), 147-156.
- [38] Tunc, E. "Instability of solutions of certain nonlinear vector differential equations of third order", *Electron. J. Differential Equations*, no. 51, (2005) 1-6 (electronic).

DEPARTMENT OF MATHEMATICS, FACULTY OF ARTS AND SCIENCES, YÜZÜNCÜ YIL UNIVERSITY
65080, VAN -TURKEY

E-mail address: cemtunc@yahoo.com