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DISCRETE SERIES REPRESENTATIONS OF p-ADIC GROUPS ASSOCIATED TO SYMMETRIC SPACES

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1. Introduction

The purpose of this paper is study the natural symmetric space analogues of various notions related to discrete series representations of a p-adic group such as Schur's orthogonality relations and formal degrees.

We study representations of the group $G = \mathbf{G}(F)$ of F-rational points of a connected, reductive F-group $\mathbf{G}$, where F is a finite extension of a field $\mathbb{Q}_p$ of p-adic numbers for some odd prime p.

The representations of interest are associated to a symmetric space $H\backslash G$, where $H = \mathbf{H}(F)$ and $\mathbf{H}$ is the group of fixed points of some F-automorphism θ of $\mathbf{G}$ of order two.

To be more precise, we are interested in irreducible admissible complex representations (π, V) of G that are H-*distinguished* in the sense that there exists a nonzero linear form $\tilde{\lambda} : V \to \mathbb{C}$ that is H-fixed or, in other words,

$$\langle \pi(h)v, \tilde{\lambda}\rangle = \langle v, \tilde{\lambda}\rangle,$$

for all $h \in H$ and $v \in V$. From now on, assume that such a representation (π, V) has been fixed. Note that H-distinction[1] implies that the restriction of the central quasi-character of π to Z_H is trivial.

The latter linear forms, together with 0, comprise the space $\mathrm{Hom}_H(\pi, 1)$ and Frobenius Reciprocity maps this space isomorphically onto the space

$$\mathrm{Hom}_G(\pi, C^\infty(H\backslash G)),$$

where $C^\infty(H\backslash G)$ is the space of smooth complex-valued functions on $H\backslash G$ viewed as a G-module with respect to right translations by G. So π is H-distinguished precisely when it has a G-invariant embedding in $C^\infty(H\backslash G)$. In this sense, the H-distinguished representations are precisely the representations that contribute to the harmonic analysis on $H\backslash G$.

For convenience, we will make some simplifying assumptions that are generally satisfied in applications. Let $(\tilde{\pi}, \widetilde{V})$ be the contragredient of (π, V). We assume that $\mathrm{Hom}_H(\tilde{\pi}, 1)$, in addition to $\mathrm{Hom}_H(\pi, 1)$, is nonzero and, furthermore, we assume that both of the latter spaces are finite-dimensional.

Let $\mathbf{Z}$ be the center of $\mathbf{G}$ and let $\mathbf{Z_H} = \mathbf{Z} \cap \mathbf{H}$ and let $Z = \mathbf{Z}(F)$ and $Z_H = \mathbf{Z_H}(F)$. We fix a Haar measure on H/Z_H for use in our integrations over the latter quotient.

[1]At the suggestion of Hervé Jacquet, we refer to the property of being H-distinguished as "H-distinction," rather than "H-distinguishedness."

Recall that (π, V) is said to be a *discrete series representation* if its central quasi-character is unitary and the absolute value of every matrix coefficient of π lies in $L^2(G/Z)$.

Definition. *If the restriction to H of every matrix coefficient of π lies in $L^1(H/Z_H)$ then π is said to be* θ**-discrete**. We now give a symmetric space analogue of this notion.

When π is θ-discrete, we may define the pairing

$$\langle v, \tilde{v}\rangle_\theta = \int_{H/Z_H} \langle \pi(h)v, \tilde{v}\rangle \; dh$$

for $v \in V$ and $\tilde{v} \in \widetilde{V}$. To appreciate the above terminology, one should consider the so-called "group case." In this case, $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_1$, for some connected reductive F-group $\mathbf{G}_1$ and let $\theta(g_1, g_2) = (g_2, g_1)$. Then if (π, V) is H-distinguished it must have the form $(\pi_1 \times \tilde{\pi}_1, V_1 \otimes \widetilde{V}_1)$. The contragredient of (π, V) is then $(\tilde{\pi}_1 \times \pi_1, \widetilde{V}_1 \otimes V_1)$ and the invariant pairing on $V \times \widetilde{V}$ is just

$$\langle v \otimes \tilde{v}, \tilde{u} \otimes u\rangle = \langle v, \tilde{u}\rangle \; \langle u, \tilde{v}\rangle.$$

Thus

$$\langle v \otimes \tilde{v}, \tilde{u} \otimes u\rangle_\theta = \int_{G_1/Z(G_1)} \langle \pi_1(g)v, \tilde{u}\rangle \; \langle u, \tilde{\pi}_1(g)\tilde{v}\rangle \; dg,$$

where $Z(G_1)$ is the center of G_1. These integrals occur in Schur's orthogonality relations when π_1 is a discrete series representation.

Definition. *If π is θ-discrete and $v \in V$ and $\tilde{v} \in \widetilde{V}$ then the function $f^\theta_{v\otimes\tilde{v}} \in C^\infty(H\backslash G)$ defined by*

$$f^\theta_{v\otimes\tilde{v}}(g) = \langle \pi(g)v, \tilde{v}\rangle_\theta$$

is called a θ**-matrix coefficient** *of π.*

Definition. *If π is θ-discrete and every θ-matrix coefficient of π is supported in a compact subset of $ZH\backslash G$ then we say that π is* θ**-supercuspidal**.

Definition. (Kato and Takano [KT]) *If the function*

$$f^\theta_{v\otimes\tilde{\lambda}}(g) = \langle \pi(g)v, \tilde{\lambda}\rangle$$

is supported in a compact subset of $ZH\backslash G$, for all $v \in V$ and all $\tilde{\lambda} \in \mathrm{Hom}_H(\pi, 1)$ then we say that π is H**-relatively cuspidal**.

If π is θ-discrete and $\tilde{v} \in \widetilde{V}$ then there is an associated invariant linear form $\tilde{\lambda}_{\tilde{v}} \in \mathrm{Hom}(\pi, 1)$ by

$$\langle v, \tilde{\lambda}_{\tilde{v}}\rangle = \langle v, \tilde{v}\rangle_\theta.$$

The following lemma follows easily from the latter fact:

Lemma. *If π is supercuspidal or H-relatively cuspidal then it is θ-supercuspidal.*

2. Formal degrees and orthogonality relations

If π is a discrete series representation then the Schur orthogonality relations hold and they say that there exists a nonzero constant $d(\pi)$ (depending on the choice of a Haar measure on G/Z) such that

$$\int_{G/Z} \langle \pi(g)v, \tilde{u}\rangle \ \langle u, \tilde{\pi}(g)\tilde{v}\rangle \ dg = d(\pi)^{-1}\langle v, \tilde{v}\rangle \ \langle u, \tilde{u}\rangle,$$

for all $u, v \in V$ and $\tilde{u}, \tilde{v} \in \widetilde{V}$. The constant $d(\pi)$ is called *the formal degree of* π (with respect to the given measure on G/Z).

It is well known that if π is a supercuspidal representation that is compactly induced from an irreducible representation ρ of an open compact-mod-center subgroup K of G then $d(\pi)$ is quotient of the degree of ρ and the measure of the image of K in G/Z. One can find a proof of this fact in [M1]. We generalize both the statement of this result and the proof later in this paper.

2.1. **The multiplicity one case.** In this section, we consider symmetric space generalizations of the formal degree and Schur's orthogonality relations. We make the simplifying assumption that the spaces $\mathrm{Hom}_H(\tilde{\pi}, 1)$ and $\mathrm{Hom}_H(\pi, 1)$ have dimension one and we fix nonzero linear forms $\lambda \in \mathrm{Hom}_H(\tilde{\pi}, 1)$ and $\tilde{\lambda} \in \mathrm{Hom}_H(\pi, 1)$.

Let ω denote the central character of the discrete series representation (π, V). Let $C^\infty(G, \omega)$ denote the space of smooth complex-valued functions f on G such that

$$f(zg) = \omega(z)^{-1} f(g),$$

for all $z \in Z$ and $g \in G$. For such f, we may define a vector $\pi(f)\lambda \in V$ by the relation

$$\langle \pi(f)\lambda, \tilde{v}\rangle = \int_{G/Z} f(g) \ \langle \lambda, \tilde{\pi}(g)^{-1}\tilde{v}\rangle \ dg,$$

for all $\tilde{v} \in \widetilde{V}$. We also define

$$\Theta_{\lambda\otimes\tilde{\lambda}}(f) = \langle \pi(f)\lambda, \tilde{\lambda}\rangle.$$

This is a linear functional on $C^\infty(G, \omega)$ and it is a generalized matrix coefficient distribution. (In fact, it is a distribution on the ℓ-sheaf $C^\infty(G, \omega)$, in the sense of [BZ].) If we take the test function f to be a matrix coefficient

$$f_{\tilde{v}\otimes v}(g) = \langle v, \tilde{\pi}(g)\tilde{v}\rangle$$

of $\tilde{\pi}$ then we have the following extension of Schur's orthogonality relations:

Lemma. ([H]) $\quad \Theta_{\lambda\otimes\tilde{\lambda}}(f_{\tilde{v},v}) = d(\pi)^{-1}\langle \lambda, \tilde{v}\rangle \ \langle v, \tilde{\lambda}\rangle.$

Proof. Fix a compact open subgroup K of G that fixes v and $\tilde{v}$. Let e_K be the convolution idempotent associated to K. Let $\lambda_K = \pi(e_K)\lambda \in V$ and $\tilde{\lambda}_K = \tilde{\pi}(e_K)\tilde{\lambda} \in \widetilde{V}$. Then

$$\begin{aligned} \Theta_{\lambda\otimes\tilde{\lambda}}(f_{\tilde{v},v}) &= \int_{G/Z} \langle \pi(g)\lambda_K, \tilde{\lambda}_K\rangle \ \langle v, \tilde{\pi}(g)\tilde{v}\rangle \ dg \\ &= d(\pi)^{-1}\langle \lambda_K, \tilde{v}\rangle \ \langle v, \tilde{\lambda}_K\rangle \\ &= d(\pi)^{-1}\langle \lambda, \tilde{v}\rangle \ \langle v, \tilde{\lambda}\rangle. \end{aligned}$$

□

Proposition. *There exists a unique nonzero constant $d_\theta(\pi)$ (depending on the choice of measure on H/Z_H used to define $\langle\ ,\ \rangle_\theta$ and on the choices of λ and $\tilde{\lambda}$) such that*

$$\langle v, \tilde{v}\rangle_\theta = d_\theta(\pi)^{-1}\langle v, \tilde{\lambda}\rangle\ \langle \lambda, \tilde{v}\rangle$$

for all θ-discrete representations (π, V) and all $v \in V$ and $\tilde{v} \in \widetilde{V}$.

Proof. Consider $\langle v, \tilde{v}\rangle_\theta$ as $\tilde{v}$ is fixed and v varies. This defines an element of $\mathrm{Hom}_H(\pi, 1)$ and hence a multiple of $\tilde{\lambda}$. Thus there exists a complex number $\gamma(\tilde{v})$ such that $\langle v, \tilde{v}\rangle_\theta = \gamma(\tilde{v})\ \langle v, \tilde{\lambda}\rangle$. Now consider $\gamma(\tilde{v})$ as $\tilde{v}$ varies. This must be a multiple of λ. Therefore, there must be a constant c such that

$$\langle v, \tilde{v}\rangle_\theta = c\ \langle v, \tilde{\lambda}\rangle\ \langle \lambda, \tilde{v}\rangle.$$

Now choose v and $\tilde{v}$ so that $\langle v, \tilde{\lambda}\rangle$ and $\langle \lambda, \tilde{v}\rangle$ are nonzero. From the previous lemma, we have

$$\Theta_{\lambda\otimes\tilde{\lambda}}(f_{\tilde{v},v}) = d(\pi)^{-1}\langle v, \tilde{\lambda}\rangle\ \langle \lambda, \tilde{v}\rangle$$

and thus $\Theta_{\lambda\otimes\tilde{\lambda}}(f_{\tilde{v},v})$ is nonzero. This implies that

$$\int_{H/Z_H} f_{\tilde{v},v}(hg)\ dh$$

is not identically zero for all $g \in G$. But the latter integral is just $\langle v, \tilde{\pi}(g)\tilde{v}\rangle$. This shows that the pairing $\langle\ ,\ \rangle_\theta$ is not identically zero. It follows that c is nonzero. Taking $d_\theta(\pi) = c^{-1}$ completes the proof. $\square$

2.2. **The group case.** Consider the group $G \times G$ with the involution $\theta(a, b) = (b, a)$. Fix an irreducible, smooth representation π of G and let $\Pi = \pi \times \tilde{\pi}$ and $\widetilde{\Pi} = \tilde{\pi} \times \pi$. Let $\lambda : \widetilde{V} \times V \to \mathbb{C}$ and $\tilde{\lambda} : V \times \widetilde{V} \to \mathbb{C}$ be the obvious canonical pairings. Then

$$\langle u \otimes \tilde{u}, \tilde{v} \otimes v\rangle_\theta = \int_{G/Z} \langle \pi(x)u, \tilde{v}\rangle\ \langle v, \tilde{\pi}(x)\tilde{u}\rangle\ dx^*.$$

In addition,

$$\begin{aligned} \langle u \otimes \tilde{u}, \tilde{\lambda}\rangle &= \langle u, \tilde{u}\rangle, \\ \langle \lambda, \tilde{v} \otimes v\rangle &= \langle v, \tilde{v}\rangle. \end{aligned}$$

Therefore,

$$d_\theta(\pi \times \tilde{\pi}) = d(\pi).$$

2.3. **The finite multiplicity case.** Suppose that $\mathrm{Hom}_H(\tilde{\pi}, 1)$ has finite dimension n_1 and is spanned by elements $\lambda_1, \ldots, \lambda_{n_1}$. Suppose that $\mathrm{Hom}_H(\pi, 1)$ has finite dimension n_2 and is spanned by elements $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{n_2}$. For fixed $\tilde{v} \in \widetilde{V}$, $\langle v, \tilde{v}\rangle_\theta$ defines an element of $\mathrm{Hom}_H(\pi, 1)$ and thus there exist numbers $c_j(\tilde{v})$ such that

$$\langle v, \tilde{v}\rangle_\theta = \sum_{j=1}^{n_2} c_j(\tilde{v})\langle v, \tilde{\lambda}_j\rangle.$$

On the other hand, for each j, it is easy to see that $c_j \in \mathrm{Hom}_H(\tilde{\pi}, 1)$ and so there exist numbers $D_\theta(\pi)_{ij}$ such that

$$c_j(\tilde{v}) = \sum_{i=1}^{n_1} D_\theta(\pi)_{ij}\langle \lambda_i, \tilde{v}\rangle.$$

Hence we have

$$\langle v, \tilde{v}\rangle_\theta = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} D_\theta(\pi)_{ij}\langle \lambda_i, \tilde{v}\rangle\langle v, \tilde{\lambda}_j\rangle.$$

2.4. **Induced supercuspidal representations.** Let K be a θ-stable subgroup of G that contains Z and is such that the quotient K/Z is compact. Let $K_H = K\cap H$. Let (ρ, W) be an irreducible (finite-dimensional) complex representation of K with unitary central character and let $(\tilde{\rho}, \widetilde{W})$ denote the contragredient.

The representations (π, V) and $(\tilde{\pi}, \widetilde{V})$ obtained by compactly-supported induction from (ρ, W) and $(\tilde{\rho}, \widetilde{W})$ are irreducible supercuspidal representations. We use the standard pairing $\langle\ ,\ \rangle$ on $W\times\widetilde{W}$ and use this to define an invariant pairing on $V\times\widetilde{V}$ as follows:

$$\langle v, \tilde{v}\rangle = \sum_{Kg\in K\backslash G}\langle v(g), \tilde{v}(g)\rangle.$$

Let $\{e_i\}$ and $\{\tilde{e}_j\}$ be bases of the spaces of K_H-fixed vectors in W and $\widetilde{W}$, respectively. Define

$$\langle w, \tilde{w}\rangle_\theta = \int_{K_H/Z_H}\langle \rho(h)w, \tilde{w}\rangle\ dh,$$

where we choose the Haar measure on K_H/Z_H so that K_H/Z_H has volume one. Given $w\in W$ and $\tilde{w}\in\widetilde{W}$, there exist unique coefficients w_i and $\tilde{w}_j$ such that

$$\int_{K_H/Z_H}\rho(h)w\ dh = \sum_i w_i e_i, \qquad \int_{K_H/Z_H}\tilde{\rho}(h)\tilde{w}\ dh = \sum_j \tilde{w}_j e_j.$$

We have

$$\langle w, \tilde{w}\rangle_\theta = \sum_{i,j} w_i\tilde{w}_j D_\theta(\rho)_{ij},$$

where

$$D_\theta(\rho)_{ij} = \langle e_i, \tilde{e}_j\rangle.$$

Given $w\in W$, there is an associated element $v_w\in V$ that is defined by $v_w(k) = \rho(k)w$, for all $k\in K$, and $v_w|(G-K)\equiv 0$. This is a K-equivariant embedding of ρ in π. By Frobenius reciprocity, ρ occurs in π with multiplicity one.

Similarly, we associate to $\tilde{w}\in\widetilde{W}$ an element $\tilde{v}_{\tilde{w}}\in\widetilde{V}$. The map $(w,\tilde{w})\mapsto(v_w, \tilde{v}_{\tilde{w}})$ is an isometric embedding in the sense that $\langle w, \tilde{w}\rangle = \langle v_w, \tilde{v}_{\tilde{w}}\rangle$. We now compute $\langle v_w, \tilde{v}_{\tilde{w}}\rangle_\theta$. We use the measure on H/Z_H given by the integral formula

$$\int_{H/Z_H} f(h)\ dh = \sum_{h\in K_H\backslash H}\int_{K_H/Z_H} f(kh)\ dk.$$

We have

$$\begin{aligned}
\langle v_w, \tilde{v}_{\tilde{w}}\rangle_\theta &= \sum_{h\in K_H\backslash H}\int_{K_H/Z_H}\langle \pi(kh)v_w, \tilde{v}_{\tilde{w}}\rangle\ dk\\
&= \int_{K_H/Z_H}\langle \pi(k)v_w, \tilde{v}_{\tilde{w}}\rangle\ dk\\
&= \int_{K_H/Z_H}\langle \rho(k)w, \tilde{w}\rangle\ dk\\
&= \langle w, \tilde{w}\rangle_\theta.
\end{aligned}$$

Therefore, in the notation of the previous section,

$$\sum_{i,j} v_i\ \tilde{v}_j\ D_\theta(\pi)_{ij} = \sum_{ij} w_i\ \tilde{w}_j\ D_\theta(\rho)_{ij},$$

where $v_i = \langle v_w, \tilde{\lambda}_j\rangle$ and $\tilde{v}_j = \langle \lambda_i, \tilde{v}_{\tilde{w}}\rangle$.

To proceed further, we now exploit the fact that all of representations are necessarily unitarizable. More precisely, we let $\overline{W}$ be the set W together with its additive structure, but with scalar multiplication defined by $c \cdot w = \bar{c}w$. Letting $\bar{\rho}(k) = \rho(k)$, we obtain a representation $(\bar{\rho}, \overline{W})$ of K. Let $(\ ,\ )$ be an invariant non-degenerate hermitian form on W that realizes ρ as a unitary representation. Then $v \mapsto \tilde{v} = (\ , v)$ defines an isomorphism of $(\bar{\rho}, \overline{W})$ with $(\tilde{\rho}, \widetilde{W})$.

We take $\{e_i\}$ to be an orthonormal basis of the space of K_H-fixed vectors in W. Then $\{\tilde{e}_i\}$ is a dual basis of the K_H-fixed vectors in $\widetilde{W}$. In particular, the spaces of K_H-fixed vectors in W and $\widetilde{W}$ have the same dimension. The matrix $D_\theta(\rho)$ is now an identity matrix. The only invariant of the matrix is its rank (which is its trace) and taking $w = e_1 + \cdots + e_r$ we have

$$\langle w, \tilde{w}\rangle_\theta = \mathrm{trace}(D_\theta(\rho)) = r = \dim W^{K_H},$$

where W^{K_H} is the space of K_H-fixed vectors in W.

Now each basis element e_i gives an element $\tilde{e}_i \in \widetilde{W}$ and this yields an element $\tilde{v}_{\tilde{e}_i} \in \widetilde{V}$. We now define $\tilde{\lambda}_i \in \mathrm{Hom}_H(\pi, 1)$ by

$$\langle v, \tilde{\lambda}_i\rangle = \langle v, \tilde{v}_{\tilde{e}_i}\rangle_\theta.$$

This linear form must be nonzero since

$$\langle v_{e_i}, \tilde{\lambda}_i\rangle = \langle v_{e_i}, \tilde{v}_{\tilde{e}_i}\rangle_\theta = \langle e_i, \tilde{e}_i\rangle_\theta = 1.$$

Define invariant linear forms λ_j similarly on $\widetilde{V}$. Suppose that the latter linear forms span the H-invariant linear forms in V and $\widetilde{V}$. Then if $i, j \in \{1, \ldots, r\}$ then

$$D_\theta(\pi)_{ij} = \langle v_{e_j}, \tilde{v}_{\tilde{e}_i}\rangle_\theta = \delta_{ij}.$$

The space V is a space of functions on G. It has a natural decomposition

$$V = \bigoplus_{KgH \in K\backslash G/H} V_{KgH},$$

where V_{KgH} is the space of functions in V that have support contained in the double coset KgH. Let V^* be the space of linear forms on V and let $(V^*)^H$ be the subspace of H-fixed linear forms. Defining V^*_{KgH} and $(V^*_{KgH})^H$ similarly, we have

$$(V^*)^H = \bigoplus_{KgH \in K\backslash G/H} (V^*_{KgH})^H.$$

We may view the elements of $(V^*_{KH})^H$ as the *primary* elements of $(V^*)^H$. It is easy to see that the H- invariant linear forms $\tilde{\lambda}_i$ constructed above from elements of $\widetilde{W}$ are primary elements, in the latter sense, and, in fact, all primary elements are obtained in this manner. Indeed, this follows from the fact that $(V^*_{KgH})^H \cong \widetilde{W}^{K\cap gHg^{-1}}$. (See [HMa1] and [HMa2].) For examples of non-primary invariant linear forms see [HM].

Proposition. *Let (π, V) be an H-distinguished supercuspidal representation that is induced from an open compact-mod-center θ-stable subgroup K. Then every H-invariant linear form $\tilde{\lambda}$ on V has the form $\tilde{\lambda}_{\tilde{v}}$ for some $\tilde{v} \in \tilde{V}$. Therefore, π must be H-relatively cuspidal.*

Proof. We may as well assume π has a unitary central character, since there is no harm in replacing π by a twist by a quasi-character. Next, we may as well assume that $\tilde{\lambda}$ is supported in a single double coset KgH. Then λ is associated to some element $\tilde{w} \in \widetilde{W}^{K\cap gHg^{-1}}$. Define $\tilde{v}$ by $\tilde{v}(kg) = \tilde{\rho}(k)\tilde{w}$, for all $k \in K$ and $\tilde{v}|(G - Kg) \equiv 0$. Then $\tilde{\lambda}_{\tilde{v}}$ is the element of $(V^*_{KgH})^H$ associated to $\tilde{w}$. Our claim follows. $\square$

3. Orbital integrals

3.1. **The Harish-Chandra/Rader/Silberger formula.** Fix an elliptic Cartan subgroup Γ of G, that is, a Cartan subgroup such that Γ/Z is compact. Suppose f is a smooth function on G with compact-mod-center support. Define

$$F_f(\gamma) = |D(\gamma)|^{1/2} \int_{G/Z} f(g\gamma g^{-1})\ dg,$$

for $\gamma \in \Gamma' = \Gamma \cap G'$. Then F_f is a smooth function on Γ'.

We now recall the derivation of the supercuspidal case of an integral formula proved by Harish-Chandra [HC] and generalized by Rader and Silberger [RS].

Assume π is irreducible supercuspidal and $\gamma \in \Gamma'$ and $f(g) = \langle \pi(g)v, \tilde{v}\rangle$.

$$\begin{aligned}
\int_{G/Z} f(g\gamma g^{-1})\ dg &= \int_{G/Z} \langle \pi(g\gamma g^{-1})v, \tilde{v}\rangle\ dg \\
&= \int_{G/Z} \sum_i \langle \pi(g\gamma g^{-1})v, \tilde{\pi}(g\gamma)\tilde{e}_i\rangle\ \langle \pi(g\gamma)e_i, \tilde{v}\rangle\ dg \\
&= \int_{G/Z} \sum_i \langle \pi(g^{-1})v, \tilde{e}_i\rangle\ \langle \pi(\gamma)e_i, \tilde{\pi}(g^{-1})\tilde{v}\rangle\ dg \\
&= \sum_i \int_{G/Z} \langle \pi(g^{-1})v, \tilde{e}_i\rangle\ \langle \pi(\gamma)e_i, \tilde{\pi}(g^{-1})\tilde{v}\rangle\ dg \\
&= \sum_i \int_{G/Z} \langle \pi(g)v, \tilde{e}_i\rangle\ \langle \pi(\gamma)e_i, \tilde{\pi}(g)\tilde{v}\rangle\ dg \\
&= d(\pi)^{-1}\langle v, \tilde{v}\rangle \sum_i \langle \pi(\gamma)e_i, \tilde{e}_i\rangle \\
&= d(\pi)^{-1}\langle v, \tilde{v}\rangle\ \Theta_\pi(\gamma),
\end{aligned}$$

where Θ_π is the character of π.

It follows that if π is irreducible supercuspidal then

$$F_f(\gamma) = d(\pi)^{-1} f(1)\ |D(\gamma)|^{1/2}\ \Theta_\pi(\gamma)$$

for all $\gamma \in \Gamma'$ and all f in the vector space $\mathcal{A}(\pi)$ spanned by the matrix coefficients of π. This formula is an instance of the classic principle of duality between orbital integrals and characters.

3.2. **The symmetric space generalization.** The group $H \times H$ acts on G by $(h_1, h_2) \cdot g = h_1 g h_2^{-1}$. Let $(H \times H)_g$ be the isotropy group of g. Then, given a function f defined at least on HgH, one can consider the integral

$$\int_{(H\times H)/(H\times H)_g} f(h_1 g h_2^{-1})\ d(h_1, h_2),$$

assuming that it is possible to choose a nonzero invariant measure for the integration. For example, if $f(g) = \langle \pi(g)v, \tilde{v}\rangle$ and $g = 1$ then the orbital integral is just $\langle v, \tilde{v}\rangle$.

We assume that γ is an element of G such that $(H \times H)_\gamma$ is compact modulo $Z_H \times Z_H$. This is the case, for example, if γ is *θ-elliptic-regular* in the sense that $\gamma\theta(\gamma)^{-1}$ is elliptic regular.

Let (π, V) be a θ-discrete representation and assume that $\mathrm{Hom}_H(\tilde{,}1)$ and $\mathrm{Hom}_H(\pi, 1)$ are 1-dimensional and are spanned by nonzero elements λ and $\tilde{\lambda}$, respectively. Let Θ be the smooth function on the θ-regular set that represents the distribution $\Theta_{\lambda,\tilde{\lambda}}$. (See [RR] for more details on this terminology.)

Let

$$f(g) = \langle \pi(g)v, \tilde{v}\rangle$$

be a matrix coefficient for π. The un-normalized orbital integral of f at a θ-elliptic-regular element γ is

$$\Phi_f(\gamma) = \int_{(H/Z_H)^2} f(h_1\gamma h_2)\ dh_1\ dh_2.$$

Proposition. $\Phi_f(\gamma) = d_\theta^{-2}(\pi)\ \langle \lambda, \tilde{v}\rangle\ \langle v, \tilde{\lambda}\rangle\ \Theta(\gamma).$

Proof. We have

$$\begin{aligned} \Phi_f(\gamma) &= \int_{(H/Z_H)^2} \langle \pi(h_1\gamma h_2)v, \tilde{v}\rangle\ dh_1\ dh_2 \\ &= \int_{H/Z_H} \langle \pi(\gamma h)v, \tilde{v}\rangle_\theta\ dh \\ &= d_\theta^{-1}(\pi)\ \langle \lambda, \tilde{v}\rangle \int_{G^\theta/Z^\theta} \langle \pi(\gamma h)v, \tilde{\lambda}\rangle\ dh. \end{aligned}$$

We now invoke the key lemma of Rader and Rallis [RR] in their proof of smoothness on the θ-regular set of spherical characters. It says that we can choose a compact open subgroup K_H of H such that

$$\int_{K_H} \pi(k\gamma^{-1})\tilde{\lambda}\ dk$$

lies in $\widetilde{V}$. Call this vector $\tilde{v}_\gamma$. Then

$$\begin{aligned} \int_{H/Z_H} \langle \pi(\gamma h)v, \tilde{\lambda}\rangle\ dh &= \int_{H/Z_H} \langle \pi(h)v, \tilde{v}_\gamma\rangle\ dh \\ &= \langle v, \tilde{v}_\gamma\rangle_\theta \\ &= d_\theta^{-1}(\pi)\ \langle v, \tilde{\lambda}\rangle\ \langle \lambda, \tilde{v}_\gamma\rangle \\ &= d_\theta^{-1}(\pi)\ \langle v, \tilde{\lambda}\rangle\ \Theta(\gamma). \end{aligned}$$

Therefore, we obtain the desired formula. □

References

[BZ] I. N. Bernstein and A. V. Zelevinskii, Representations of the group GL(n, F) where F is a non-archimedean local field, Russian Math. Surveys 31:3 (1976), 1–68.

[H] J. Hakim, Supercuspidal Gelfand pairs, J. Number Theory **100** (2003), 251–269.

[HMa1] J. Hakim and Z. Mao, Cuspidal representations associated to $(GL(n), O(n))$ over finite fields and p-adic fields, J. Algebra **213** (1999), 129–143.

[HMa2] J. Hakim and Z. Mao, Supercuspidal representations of $GL(n)$ distinguished by a unitary subgroup, Pacific J. Math **185** (1998), no. 1, 149–162.

[HMu] J. Hakim and F. Murnaghan, Distinguished tame supercuspidal representations, preprint.

[HC] Harish-Chandra, A submersion principle and its applications, Papers dedicated to the memory of V. K. Patodi, Indian Academy of Sciences, Bangalore, and the Tata Institute of Fundamental Research, Bombay, 1980, pp. 95–102.

[KT] S.-I. Kato and K. Takano, Subrepresentation theorem for p-adic symmetric spaces, preprint.

[M1] F. Murnaghan, Representations of reductive p-adic groups, Fall 1997, unpublished course notes.

[M2] F. Murnaghan, Spherical characters: the supercuspidal case, Con. Math., to appear.

[RR] C. Rader and S. Rallis, Spherical characters on $\mathfrak{p}$-adic symmetric spaces, Amer. J. Math. 118 (1996), 91–178.

[RS] C. Rader and A. Silberger, Some consequences of Harish-Chandra's submersion principle, Proc. Amer. Math. Soc. 118 (1993), no. 4, 1271–1279.

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