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POLYNOMIAL COMPLEXITY FOR HILBERT SERIES OF BOREL TYPE IDEALS

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Abstract. In this paper, it is shown that the Hilbert series of a Borel type ideal may be computed within a complexity which is polynomial in D^n where $n+1$ is the number of unknowns and D is the highest degree of a minimal generator of input (monomial) ideal.

1. Introduction

A classical algorithm to compute the Hilbert series of a monomial ideal, is from its free resolution which is infeasible in practice. Bayer and Stillman [2] have proved that the computation of Hilbert series of a monomial ideal is at least difficult as an NP-complete problem in the number of variables, see also [5]. For some class of monomial ideals, the computation of the Hilbert series may be less costly than NP-complete. For example, Bayer and Stillman [2] have shown that the Hilbert series of a Borel monomial ideal may be computed in linear time. Recall that a monomial ideal J over the ring $R = K[x_0, \ldots, x_n]$ where K is an arbitrary field is defined to be Borel if $x_j m \in J$ implies that $x_i m \in J$ for any $i < j$.

In this paper, we study the complexity of computing the Hilbert series of a *Borel type ideal*. A monomial ideal $J \subset R$ is Borel type if it satisfies the following property:

$$J : x_j^\infty = J : \langle x_0, \ldots, x_j \rangle^\infty$$

for all $j = 1, \ldots, n$ (see [1, 12]). *We show that the Hilbert series of a Borel type ideal may be computed within a complexity which is polynomial in D^n where $n+1$ is the number of unknowns and D is the highest degree of a minimal generator of input polynomials.* For this, we describe an algorithm to decide within the same complexity whether a monomial ideal is Borel type or not. Also, we establish a sharper upper bound for the satiety and Castelnuovo-Mumford regularity of such an ideal and we prove that these invariants may be computed within the above complexity. Finally, as an application of our results, we give a new formula to compute the degree of a Borel type ideal. This paper is a continuation of the ideas which have first appeared in [11].

It is well-known that to compute the Hilbert series of a general ideal, we reduce it to a monomial ideal by Gröbner basis computation. On the other hand, it follows from the work of Mayr and Meyer [14] that the problem of computing a Gröbner basis (in worst case) is exponential space complete. Both cardinality and maximal

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degree of a Gröbner basis might be doubly exponential in the number of variables. Our result shows that (for an ideal whose initial ideal is Borel type) if the problem of computing a Gröbner basis is simple then that of Hilbert series is not more difficult. In fact, our computation suggests that the expensive part of computing the Hilbert series of a general ideal is the Gröbner basis computation. This leads us to the following conjecture which generalizes our result.

Conjecture 1.1. *The Hilbert series of a monomial ideal may be computed within a complexity which is polynomial in D^n where $n+1$ is the number of unknowns and D is the highest degree of a minimal generator of the input ideal.*

The main interest of our algorithm is its bound of complexity. In fact, with the existing implementations (see [5, 2]), the computation of the Hilbert series is negligible with respect to that of the Gröbner basis which is needed for. It is therefore not worthwhile to spent human time to efficiently implement our algorithm.

Now, we give the structure of the paper. In Sections 2, we recall the definition of a Borel type ideal and we describe a polynomial-time algorithm to test whether a monomial ideal is Borel type or not. In Section 3 (resp. 4) we establish an upper bound and describe an algorithm having the complexity polynomial in D^n for the satiety (resp. Castelnuovo-Mumford regularity) of a Borel type ideal. In Section 5, we prove that one can compute the Hilbert series of a Borel type ideal within this complexity. In Section 6, we give a formula to compute the degree of a Borel type ideal. Finally, Section 7 presents our conclusions.

2. Borel type ideals

The purpose of this section is to study a certain class of monomial ideal which we call *Borel type ideals*. We describe also an algorithm which determines whether a monomial ideal is Borel type or not within a complexity which is polynomial in input size.

We recall first the definition of the saturation of a homogeneous ideal. Let J be a monomial ideal of the polynomial ring $R = K[x_0, \ldots, x_n]$ where K is an arbitrary field. If $\mathfrak{m} = \langle x_0, \ldots, x_n \rangle$ is the unique maximal homogeneous ideal of R, then we recall that the ideal $J : \mathfrak{m}^i$ is defined for any positive integer i as

$$J : \mathfrak{m}^i = \{F \in R \mid \forall G \in \mathfrak{m}^i,\ GF \in J\}.$$

The ideal $J : \mathfrak{m}^\infty$ is defined $\bigcup_{i=1}^{\infty} J : \mathfrak{m}^i$. Denote by J_ℓ the set of homogeneous elements of degree ℓ of J.

Proposition 2.1. *The ideal $J^{\mathrm{sat}} = J : \mathfrak{m}^\infty$ is called the* saturation *of J. It is the unique largest ideal $I \subset R$ for the following property:*

$$\exists s \text{ such that } \forall \ell \geq s \quad I_\ell = J_\ell.$$

Proof. One can check easily that the saturation of J satisfies this property. □

For a monomial ideal J, we introduce the following sequences of ideals associated to J. Let $R_i = K[x_0, \ldots, x_i]$.

Notation 2.2. *Let* $\mathrm{sec}(J, 0) = \overline{\mathrm{sec}}(J, 0) = J$ *and for* $i = 1, \ldots, n+1$*:*

- $\mathrm{sec}(J, i) = J + \langle x_{n-i+1}, \ldots, x_n \rangle$.
- $\overline{\mathrm{sec}}(J, i) = \mathrm{sec}(J, i) \cap R_{n-i} = J|_{x_{n-i+1} = \cdots = x_n = 0} \cap R_{n-i}$.

Note that $\mathrm{sec}(J, i)$ and $\overline{\mathrm{sec}}(J, i)$ are ideals of R and R_{n-i} respectively for any i.

Lemma 2.3. *Let $J \subset R$ be a monomial ideal. For any $\ell \leq n$, the following conditions are equivalent:*

(1) $\sec(J,i)^{\mathrm{sat}} = \sec(J,i) : x_{n-i}^{\infty}$ *for* $i = 0, \ldots, \ell$.
(2) $\overline{\sec}(J,i)^{\mathrm{sat}} = \overline{\sec}(J,i) : x_{n-i}^{\infty}$ *for* $i = 0, \ldots, \ell$.
(3) $J : \langle x_0, \ldots, x_{n-i} \rangle^{\infty} = J : x_{n-i}^{\infty}$ *for* $i = 0, \ldots, \ell$.

Proof. (1) $\Rightarrow$ (3). We proceed by induction on i. For $i = 0$, we have $I^{\mathrm{sat}} = I : x_n^{\infty}$ by definition of sec, and this proves the assertion in this case. Suppose that the assertion is true for $i-1$. We have to prove that any (monomial) minimal generator $m \in J : x_{n-i}^{\infty}$ belongs to $J : \langle x_0, \ldots, x_{n-i} \rangle^{\infty}$. For some k, we have $x_{n-i}^k m \in J \subset \sec(J,i)$. Thus, $m \in \sec(J,i)^{\mathrm{sat}}$ by (1), and therefore $x_j^t m \in \sec(J,i) = J + \langle x_{n-i+1}, \ldots, x_n \rangle$ for some integer t and for any j. We claim that $m \notin \langle x_{n-i+1}, \ldots, x_n \rangle$. If this claim is true, $x_j^t m \in J$ for $j = 0, \ldots, n-i$, and this proves the assertion.

Proof of the claim: We prove it ad absurdum. Let $m = x_j m'$ for some $j \in \{n-i+1, \ldots, n\}$. Thus $m' \in J : x_j^{\infty} = J : \langle x_0, \ldots, x_j \rangle^{\infty}$ by the hypothesis of induction. This implies that $m' \in J : x_{n-i}^{\infty}$. Since m is a minimal generator of $J : x_{n-i}^{\infty}$, this is impossible.

(3) $\Rightarrow$ (1). It is enough to prove that any monomial $m \in \sec(J,i) : x_{n-i}^{\infty}$ belongs to $\sec(J,i)^{\mathrm{sat}}$ for any i. Two cases are possible: If m belongs to $\langle x_{n-i+1}, \ldots, x_n \rangle$ then there is nothing to prove because $\langle x_{n-i+1}, \ldots, x_n \rangle \subset \sec(J,i)^{\mathrm{sat}}$. If not, there exists an integer k such that $x_{n-i}^k m \in J$. This implies that $m \in J : x_{n-i}^{\infty} = J : \langle x_0, \ldots, x_{n-i} \rangle^{\infty}$ by (3). Thus, there exists an integer t such that $x_j^t m \in J$ for $j = 0, \ldots, n-i$, and therefore $x_j^t m \in \sec(J,i)$ for any j. This implies that $m \in \sec(J,i)^{\mathrm{sat}}$. This argument was inspired by the proof of [4], Proposition 3.2.

(2) $\Rightarrow$ (3). The proof is similar to (1) $\Rightarrow$ (3).

(3) $\Rightarrow$ (2). It suffices to show that any monomial $m \in \overline{\sec}(J,i) : x_{n-i}^{\infty}$ belongs to $\overline{\sec}(J,i)^{\mathrm{sat}}$ for any i. We have $x_{n-i}^k m \in J$ for some integer k. Thus, $m \in J : x_{n-i}^{\infty} = J : \langle x_0, \ldots, x_{n-i} \rangle^{\infty}$ by (2) which implies that there exists an integer t such that $x_j^t m \in J$ for $j = 0, \ldots, n-i$. The membership $x_j^t m \in R_{n-i}$ proves the assertion. □

We recall that the dimension $\dim(J)$ of the ideal J is the dimension of the corresponding quotient ring.

Lemma 2.4. *If any condition of Lemma 2.3 is true for $\ell = \dim(J) - 1$, it is true for any ℓ.*

Proof. By Lemma 2.3, it is enough to prove the assertion for the first condition. Notice that if X is a zero-dimensional (monomial) ideal then $X^{\mathrm{sat}} = X : x_i^{\infty}$ for any i. Now apply this for $X = \sec(J,i)$ which is zero-dimensional by definition. □

Definition 2.5. *A monomial ideal $J \subset R$ is called a* Borel type ideal *if it satisfies one of the equivalent conditions in Lemma 2.3 for $\ell = \dim(J) - 1$.*

From Lemma 2.3(3), we conclude that the notions Borel type and nested type (introduced in [4]) coincide.

Lemma 2.3(2) provides a new characterization of Borel type ideals from which we derive a simple test for determining whether a monomial ideal is Borel type or not (see the following).

Algorithm testing Borel type ideal
Input: $J \subset R$ a monomial ideal
Output: The answer to "Is J a Borel type ideal?"
$G := \{m_1, \ldots, m_k\}$ a minimal system of generators for J
$Deg := \max\{\deg(m_1), \ldots, \deg(m_k)\}$
$e :=$ highest integer ℓ such that $x_i^{Deg} \in J$ for $i = 0, \ldots, \ell$
$d := n - e$
For each monomial $x_0^{e_0} \cdots x_h^{e_h} \in G$ with $h > n - d$ and $e_h > 0$ **do**
For $j = 1, \ldots, h-1$ **do**
If $x_0^{e_0} \cdots x_{h-1}^{e_{h-1}} x_j^{Deg} \notin J$ **then**
Return "No"
Return "Yes, and the dimension of the ideal is d"

Proof. (Algorithm) The termination of the algorithm is obvious. Let us show its correctness. For this, we have to prove that J is Borel type if and only if the response of the algorithm is "Yes". Suppose that J is Borel type and $x_0^{e_0} \cdots x_h^{e_h} \in G$ for some $h > n - d$ with $e_h > 0$. This implies that (Lemma 2.3(2))

$$x_0^{e_0} \cdots x_{h-1}^{e_{h-1}} \in J : x_h^\infty = J : \langle x_0, \ldots, x_h \rangle^\infty.$$

Thus, $x_0^{e_0} \cdots x_{h-1}^{e_{h-1}} x_j^{Deg}$ must be in J for any $j = 0, \ldots, h-1$, and the answer is "Yes". In this case, d is the dimension of J by Nœther normalization test (see [3], Lemma 3.1). Conversely, we can conclude that $J : x_h^\infty \subset J : \langle x_0, \ldots, x_h \rangle^\infty$, and therefore J is Borel type (Lemma 2.3(2)).

□

Remark 2.6. *The condition $h > n - d$ in this algorithm is not essential, because $x_i^{Deg} \in J$ for $i = 0, \ldots, n - d$.*

Remark 2.7. *The integer d is the dimension of J if the answer of the algorithm is "Yes" (see the proof of algorithm).*

Proposition 2.8. *The complexity of this algorithm is polynomial in kn.*

Proof. One can easily see that the number of operations in two loops "For" is $k^2 n^2$. Thus the complexity of the algorithm is polynomial in kn. □

3. Satiety of Borel type ideals

In this section, we prove a new upper bound for the satiety of a Borel type ideal, and we describe an algorithm of polynomial complexity in input size that computes the satiety of such an ideal. Let us define the satiety of a monomial ideal.

Definition 3.1. *The* satiety *of a monomial ideal $J \subset R$, denoted by* $\operatorname{sat}(J)$*, is the smallest positive integer s such that $J_\ell^{\mathrm{sat}} = J_\ell$ for all $\ell \geq s$.*

We show first an upper bound for the satiety of a Borel type ideal. For this, a lemma from [1] is needed. Here, a linear form $y \in R$ is generic for J if y is a non-zero divisor in R/J^{sat}.

Lemma 3.2. *Let $J \subset R$ be a monomial ideal and $y \in R$ be a linear form. The following conditions are equivalent:*

(1) $(J : y)_\ell = J_\ell$ *for any* $\ell \geq s$.

(2) $\operatorname{sat}(J) \leq s$ *and* y *is generic for* J.

Corollary 3.3. *Let* $J \subset R$ *be a Borel type ideal. Then*

$$\operatorname{sat}(J) = \max_{m \in (J:x_n)\setminus J} \{\deg(m)\} + 1.$$

Proof. Since x_n is generic for J from hypothesis, $(J : x_n)_\ell = J_\ell$ for any $\ell \geq \operatorname{sat}(J)$ by Lemma 3.2. Thus, the satiety of J is equal to the maximum degree of $(J : x_n)\setminus J$ plus one which proves the assertion.

□

The following theorem may follow from [4], Corollary 2.6. We give here a simpler proof for it.

Theorem 3.4. *Let* $J \subset R$ *be a Borel type ideal and let* $x_0^{D_0} \cdots x_n^{D_n}$ *be the least common multiple of the minimal generators of* J. *Then,*

$$\operatorname{sat}(J) \leq \max\{0, D_0 + \cdots + D_n - n\}.$$

Proof. Two cases are possible: If $D_0 + \cdots + D_n - n < 0$, there is some i such that $D_i = 0$. We claim that $\operatorname{sat}(J) = 0$. For this it is enough to show that $J : \mathfrak{m} = J$, i.e. J is saturated. Let $m \in J : \mathfrak{m}$ be a monomial. Thus, $x_i m \in J$ and this implies that $m \in J$ because x_i does not appear in the generators of J and this proves the claim. In the other case, by Corollary 3.3, it suffices to prove that any monomial $m \in J : x_n$ of degree $D_0 + \cdots + D_n - n$ belongs to J. From degree of m, one can show that $x_i^{D_i}$ divides m for some i. The membership $x_n m \in J$ implies that $x_i^t m \in J$ for some t because J is Borel type. This follows that $m \in J$ by the fact that $x_i^{D_i}$ divides m and D_i is the maximal degree ℓ such that x_i^ℓ appears in the minimal generators of J. □

Example 3.5. Computing the upper bounds for the satieties of some Borel type ideals. *Let* R *be the ring* $K[x_0, x_1, x_2, x_3, x_4]$. *Consider the monomial ideal* $J = \langle x_0, x_1 \rangle$. *Since* $1+1-4 < 0$, *then* $\operatorname{sat}(J) = 0$. *The satiety of* $J = \langle x_0^2, x_1^4, x_2^5, x_3^3, x_4 \rangle$ *is less than or equal to* 11 *because* $2 + 4 + 5 + 3 + 1 - 4 \geq 0$.

The following lemma is the basis for the occurrence of D^n in our complexity bounds.

Lemma 3.6. *The number of monomials of degree at most* $\delta = (n+1)(D-1)+1$ *in* $n+1$ *variables is bounded above by* $(eD)^{n+1}$ *for* D *and* $n \geq 0$.

Proof. The number of monomials of degree at most δ in $n+1$ variables is equal to $\binom{n+1+\delta}{n+1}$ (see [6] p. 106 for example). By definition of the binomial coefficients, we have

$$\begin{aligned} \binom{n+1+\delta}{n+1} &= \binom{(n+1)D+1}{n+1} \\ &= \frac{D^{n+1}}{(n+1)!} \prod_{i=1}^{n+1} \left(n+1+\frac{2-i}{D} \right). \end{aligned}$$

As $n + \frac{2-i}{D} \leq n$ for $i \geq 2$ and $\left(n+1-\frac{1}{D}\right)\left(n+1+\frac{1}{D}\right) < (n+1)^2$, we have for $n > 1$

$$\binom{n+1+\delta}{n+1} < \frac{(n+1)^{n+1}}{(n+1)!} D^{n+1},$$

which implies that $\binom{n+1+\delta}{n+1} < (eD)^{n+1}$ by Stirling's formula where $e = 2.71828\cdots$ is the usual Euler constant. For $n = 0$ the result is easily proved directly. $\square$

Corollary 3.7. *The satiety of a Borel type ideal may be computed by a complexity polynomial in D^n where D is the highest degree of its minimal generator.*

Proof. Let $J \subset R$ be a Borel type ideal. If $D = 1$, two cases are possible: If $D_n = 1$ then $J = \mathfrak{m}$ and $\operatorname{sat}(J) = 1$. In the other case, i.e. $D_n = 0$, we have $\operatorname{sat}(J) = 0$ (see Theorem 3.4). Thus, if $D = 1$ the bound polynomial in D^n holds. Now, suppose that $D \geq 2$. By Corollary 3.3 and Theorem 3.4, it is enough to find the maximal degree $h \leq \delta = (n+1)(D-1)+1$ such that there is a monomial m of this degree with $m \in (J : x_n) \setminus J$. The number of these monomials is $(eD)^{n+1}$ (Lemma 3.6) and the cost of the last condition is $k(n+1)$ operations where k is the number of minimal generators of J. Thus the complexity of this computation is $k(n+1)(eD)^{n+1}$. This is polynomial in D^n because $k \leq (eD)^{n+1}$ (by Lemma 3.6), $n+1 \leq 2^{0.55(n+1)} \leq D^{0.55(n+1)}$ and $e < D^{1.45}$ (by $D \geq 2$). $\square$

4. Castelnuovo-Mumford regularity of Borel type ideals

In this section, we prove a new upper bound for the Castelnuovo-Mumford regularity of a Borel type ideal, and we describe an algorithm having the polynomial complexity in input size to compute the Castelnuovo-Mumford regularity of such an ideal. Let us define the *Castelnuovo-Mumford regularity* of a monomial ideal $J \subset R$. If

$$0 \longrightarrow \bigoplus_j R(e_{rj}) \longrightarrow \cdots \longrightarrow \bigoplus_j R(e_{1j}) \longrightarrow \bigoplus_j R(e_{0j}) \longrightarrow J \longrightarrow 0$$

is a minimal graded free resolution of J, $\operatorname{reg}(J)$ is the maximal of $e_{ij} - i$ for each i and j. To establish an upper bound for the Castelnuovo-Mumford regularity of a Borel type ideal, we use the following lemmas from [1].

Lemma 4.1. *Let $J \subset R$ be a monomial ideal, and $y \in R$ be a generic linear form for J. The following conditions are equivalent:*

- $\operatorname{reg}(J) \leq s$.
- $\operatorname{sat}(J) \leq s$ *and* $\operatorname{reg}(J + \langle y \rangle) \leq s$.

Lemma 4.2. *Let $J \subset R$ be a zero–dimensional monomial ideal. The following conditions are equivalent:*

- $\operatorname{sat}(J) \leq s$.
- $\operatorname{reg}(J) \leq s$.
- *J_s is equal to the set of homogeneous polynomials of degree s of R.*

Corollary 4.3. *With the same hypothesis, we have* $\operatorname{reg}(J) = \operatorname{sat}(J)$.

Proposition 4.4. *Let $J \subset R$ be a Borel type ideal and let $d = \dim(J)$.*

$$\operatorname{reg}(J) = \max_{0 \leq i \leq d} \{\operatorname{sat}(\operatorname{sec}(J, i))\} \tag{1}$$

$$= \max_{0 \leq i \leq d} \{\operatorname{sat}(\overline{\operatorname{sec}}(J, i))\}. \tag{2}$$

Proof. To prove the equality (1), from Lemma 4.1 we have

$$\operatorname{reg}(J) = \max\{\operatorname{sat}(J), \operatorname{reg}(\operatorname{sec}(J, 1))\}.$$

By reusing this formula for the ideal $\sec(J,1)$ and using the fact that x_{n-1} is generic for $\sec(J,1)$ we obtain

$$\begin{aligned}\operatorname{reg}(J) &= \max\{\operatorname{sat}(J), \max\{\operatorname{sat}(\sec(J,1)), \operatorname{reg}(\sec(J,2))\}\}\\ &= \max\{\operatorname{sat}(J), \operatorname{sat}(\sec(J,1)), \operatorname{reg}(\sec(J,2))\}.\end{aligned}$$

So by induction, we can conclude that $\operatorname{reg}(J)$ is equal to

$$\max\{\operatorname{sat}(J), \operatorname{sat}(\sec(J,1)), \ldots, \operatorname{sat}(\sec(J,d-1)), \operatorname{reg}(\sec(J,d))\}.$$

Since $\sec(J,d)$ is zero–dimensional (see for example [10], Lemma 5), then

$$\operatorname{reg}(\sec(J,d)) = \operatorname{sat}(\sec(J,d)),$$

(Corollary 4.3), and this proves the assertion.

Let us prove (2). It is enough to show that $\operatorname{sat}(\overline{\sec}(J,i)) = \operatorname{sat}(\sec(J,i))$ for any i. By the membership $x_{n-i+1}, \ldots, x_n \in \sec(J,i)$ we have

$$\overline{\sec}(J,i)^{\operatorname{sat}} = \sec(J,i)^{\operatorname{sat}}|_{x_{n-i+1}=\cdots=x_n=0} \cap R_{n-i},$$

by definition of the saturation of an ideal, and this proves the assertion.

□

As a consequence of Proposition 4.4 and Corollary 3.7 we have:

Corollary 4.5. *The Castelnuovo-Mumford regularity of a Borel type ideal may be computed by a complexity polynomial in D^n where D is the highest degree of its minimal generator.*

If $D = 1$, we can conclude simply that the regularity of the ideal is 1 (see the proof of Corollary 3.7). Imran and Sarfraz [13], have proved the upper bound $(n+1)D - n$ for the Castelnuovo-Mumford regularity of a Borel type ideal where D is the highest degree of a minimal generator of the ideal. We improve this bound in the following theorem.

Theorem 4.6. *Let $J \subset R$ be a Borel type ideal and let $x_0^{D_0} \cdots x_n^{D_n}$ be the least common multiple of the minimal generators of J. Then,*

$$\operatorname{reg}(J) \leq \max\{D_0 + \cdots + D_{n-d} - (n-d), \ldots, D_0 + \cdots + D_n - n\}.$$

Proof. Since $\overline{\sec}(J,i)$ for any i is Borel type, then its satiety is at most $\max\{0, D_0 + \cdots + D_{n-i} - n + i\}$ by Theorem 3.4. Thus, the assertion follows from Proposition 4.4 and the fact that $D_0 + \cdots + D_{n-d} > n - d$. □

Example 4.7 (Computing an upper bound for the Castelnuovo-Mumford regularity of a Borel type ideal.)**.** *Let R be the ring $K[x_0, x_1, x_2, x_3, x_4]$. Consider the monomial ideal $J = \langle x_0, x_1^2 \rangle$. Thus, its regularity is at most* $\max\{1+2-4, 1+2-3, 1+2-2, 1+2-1\} = 2$.

5. Hilbert series of Borel type ideals

In this section, we describe an algorithm to compute the Hilbert series of a Borel type ideal within a polynomial complexity in input size.

Let X be a graded module or an ideal and δ be a positive integer. We denote by X_δ (resp. $X_{\geq\delta}$) the set of elements of X of degree (resp. at least) δ. Recall that the *Hilbert series* of a monomial ideal $J \subset R$ is the power series $\operatorname{HS}_J(t) =$

$\sum_{s=0}^{\infty} \mathrm{HF}_J(s)t^s$ where $\mathrm{HF}_J(s)$, the Hilbert function of J, is the dimension of $(R/J)_s$ as a K-vector space. From this definition, we have

$$\mathrm{HS}_{J_{<\delta}}(t) = \mathrm{HF}_J(0) + \cdots + \mathrm{HF}_J(\delta-1)t^{\delta-1} + \sum_{i=\delta}^{\infty} \binom{n+i}{n} t^i \tag{3}$$

$$\mathrm{HS}_{J_{\geq\delta}}(t) = \sum_{i=0}^{\delta-1} \binom{n+i}{n} t^i + \mathrm{HF}_J(\delta)t^{\delta} + \mathrm{HF}_J(\delta+1)t^{\delta+1} + \cdots \tag{4}$$

for any δ. Thus, we can conclude that $\mathrm{HS}_J = \mathrm{HS}_{J_{<\delta}} + \mathrm{HS}_{J_{\geq\delta}} - \mathrm{HS}_{\langle 0\rangle}$. Therefore, to prove that the Hilbert series of a Borel type ideal J may be computed within a complexity polynomial in D^n, it is enough to prove the same for these three Hilbert series for $\delta = D_0 + \cdots + D_n - n$. Let us to compute $\mathrm{HS}_{J_{\geq\delta}}$. For this, we prove first that for any monomial ideal $J \subset R$, the homogeneous ideal $J_{\geq\delta}$ is *stable*. A monomial ideal $J \subset R$ is called stable (see [7]) if for all monomial $m \in J$ we have $x_j m/x_\ell \in J$ for all $j < \ell$ where ℓ is the maximal integer i such that x_i divides m. We remark that this result has proved by Imran and Sarfraz [13] for $\delta = (n+1)D - n$ and we generalize it here by another approach.

Lemma 5.1. *Let $J \subset R$ be a Borel type ideal and let $x_0^{D_0} \cdots x_n^{D_n}$ be the least common multiple of the minimal generators of J. Then $J_{\geq\delta}$ is stable for $\delta = \max\{D_0 + \cdots + D_i - i \mid i = n-d, \ldots, n\}$ with $d = \dim(J)$.*

Proof. Let $m \in J_{\geq\delta}$ be a monomial, ℓ be the maximal integer i such that x_i divides m and $j < \ell$ be an integer. By definition $m \in \overline{\mathrm{sec}}(J, n-\ell)$. It follows by the definition of Borel ideal that $m/x_\ell \in \overline{\mathrm{sec}}(J, n-\ell) : x_{n-\ell}^{\infty} \subset \overline{\mathrm{sec}}(J, n-\ell)^{\mathrm{sat}}$. This implies that $x_j m/x_\ell \in \overline{\mathrm{sec}}(J, n-\ell)$ because $\deg(x_j m/x_\ell) \geq \delta$ is greater than $\mathrm{sat}(\overline{\mathrm{sec}}(J, n-\ell))$ by Theorem 4.6 and Proposition 4.4. Therefore $x_j m/x_\ell \in J$. Remark that for $\ell < n-d$ we used the fact that the ideal $\overline{\mathrm{sec}}(J, n-\ell)$ contains $\overline{\mathrm{sec}}(J, d)$ which is zero-dimensional and thus $\mathrm{sat}(\overline{\mathrm{sec}}(J, n-\ell)) \leq \mathrm{sat}(\overline{\mathrm{sec}}(J, d)) \leq \delta$. $\square$

Recall that $\mathrm{HS}_J(t) = P(t)/(1-t)^{n+1}$ where $P(t)$ is a polynomial in t (see [8], Theorem 7 of Chapter 11). We denote this polynomial by $\mathrm{NHS}_J(t)$.

Lemma 5.2. *With the hypotheses of Lemma 5.1, we have $\mathrm{NHS}_{J_{\geq\delta}}(t) = 1 - t^{\delta} \sum_{i=0}^{n} a_i (1-t)^i$ where a_i is the number of monomial $m \in J_\delta$ such that i is the maximal integer ℓ with $x_\ell \mid m$.*

Proof. Let $J_{\geq\delta} = \langle m_1, \ldots, m_k\rangle$ and m_i be arranged such that m_{i+1} is greater than m_i for all i using the lexicographic order $x_n > x_{n-1} > \cdots > x_0$. Thus, by [2] Corollary 2.3 we have

$$\mathrm{NHS}_{J_{\geq\delta}}(t) = \mathrm{NHS}_{\langle m_1\rangle}(t) + \sum_{i=2}^{k} t^{\deg(m_i)} \mathrm{NHS}_{\langle m_1,\ldots,m_{i-1}\rangle : m_i}(t).$$

Notice that by the stable property of $J_{\geq\delta}$ (Lemma 5.1) we have that $\langle m_1, \ldots, m_{i-1}\rangle : m_i = \langle x_0, \ldots, x_{v_i-1}\rangle$ where v_i is the maximal integer ℓ with $x_\ell \mid m_i$. Therefore, using the fact that $\deg(m_i) = \delta$

$$\mathrm{NHS}_{J_{\geq\delta}}(t) = \mathrm{NHS}_{\langle m_1\rangle}(t) + t^{\delta} \sum_{i=2}^{k} \mathrm{NHS}_{\langle x_0,\ldots,x_{v_i-1}\rangle}(t).$$

The zero-dimensionality of $J + \langle x_n, \ldots, x_{n-\dim(J)+1}\rangle$ implies that $m_1 = x_0^\delta$, and therefore $\mathrm{NHS}_{\langle m_1\rangle}(t) = 1-t^\delta$. Thus, the assertion follows from $\mathrm{NHS}_{\langle x_0,\ldots,x_{v_i-1}\rangle}(t) = (1-t)^{v_i}$ (see [2] Corollary 2.5) and definition of a_i. □

Theorem 5.3. *Let $J \subset R$ be a Borel type ideal. The Hilbert series of J may be computed by a complexity polynomial in D^n where D is the highest degree of its minimal generator.*

Proof. If $D = 1$ then $J = \langle x_0, \ldots, x_i\rangle$ for some integer i (by definition), and $\mathrm{HS}_J(t) = 1/(1-t)^{n-i}$. Thus, the bound polynomial in D^n holds in this case. Now, let $D \geq 2$ and $\delta = \max\{D_0 + \cdots + D_i - i \mid i = n - \dim(J), \ldots, n\}$ where $x_0^{D_0} \cdots x_n^{D_n}$ is the least common multiple of the minimal generators of J. Using the formula $\mathrm{HS}_J = \mathrm{HS}_{J_{<\delta}} + \mathrm{HS}_{J_{\geq\delta}} - \mathrm{HS}_{\langle 0\rangle}$ it is enough to prove the assertion for these three Hilbert series. We know that $\mathrm{HS}_{\langle 0\rangle}(t) = 1/(1-t)^{n+1}$. To compute $\mathrm{HS}_{J_{<\delta}}$ and $\mathrm{HS}_{J_{\geq\delta}}$ (Lemma 5.2) it suffices to list the monomials of degree $\leq \delta$ which are not in J. Since the number of monomials of degree $\leq \delta \leq (n+1)D - n$ is at most $(eD)^{n+1}$ and the cost of testing whether a monomial belongs to J or not is $k(n+1)$ operations, these Hilbert series is computed by $k(n+1)(eD)^{n+1}$ operations which is polynomial in D^n. □

Example 5.4. Computing the Hilbert series of a Borel type ideal. *Let R be the ring $K[x_0, x_1, x_2, x_3, x_4]$. Consider the following monomial ideal from* [4]*, Example 3.13*

$$J = \langle x_0^4, x_0^3x_1, x_0^2x_1^2, x_1^4, x_0^3x_2, x_0^2x_2^2, x_1^3x_2^5, x_0^3x_3, x_0^3x_4^2\rangle.$$

This is an ideal of dimension 3*, and it is Borel type by algorithm testing Borel type ideal. By formula* (3) *we have $\mathrm{HS}_{J_{<\delta}}(t) = P(t)/(1-t)^5$ where $\delta = 12$ and $P(t)$ is*

$$1+3t^8-6t^7+2t^6+7t^5-7t^4+t^{10}-t^{11}+1271t^{12}-4613t^{13}+6327t^{14}-3883t^{15}+899t^{16}.$$

By a simple computation (using the software [9]*) we have $a_0 = 1, a_1 = 12, a_2 = 72, a_3 = 187$ and $a_4 = 899$ (see the notation of Lemma 5.2). Thus, $\mathrm{HS}_{J_{\geq\delta}}(t)$ is equal to*

$$\frac{1 - t^{12}(13 - 12t + 72(1-t)^2 + 287(1-t)^3 + 899(1-t)^4)}{(1-t)^5}$$

which follows that

$$\begin{aligned}\mathrm{HS}_J(t) &= \mathrm{HS}_{J_{<\delta}} + \mathrm{HS}_{J_{\geq\delta}} - \frac{1}{(1-t)^5}\\ &= \frac{1 + 2t + 3t^2 + 4t^3 - 2t^4 - t^5 + 2t^6 - t^7 - t^8 - t^9}{(1-t)^3}.\end{aligned}$$

6. Degree of Borel type ideals

In this section, we give a formula for the degree of a Borel type ideal. We recall first the definition of the degree of a monomial ideal $J \subset R$.

Proposition 6.1. *We have $\mathrm{HS}_J(t) = N(t)/(1-t)^d$ where $N(t)$ is a polynomial which is not multiple of $1-t$, and $d = \dim(J)$.*

For the proof of this proposition see [8], Theorem 7 of Chapter 11. Using this proposition we define the degree of a monomial ideal.

Definition 6.2. *The degree of a monomial ideal $J \subset R$, noted by $\deg(J)$, is $N(1)$ where N is the numerator of HS_J.*

Let J be a Borel type ideal and $\delta = D_0 + \cdots + D_n - n$ where $x_0^{D_0} \cdots x_n^{D_n}$ is the least common multiple of the minimal generators of J. Using the formula $\mathrm{HS}_J = \mathrm{HS}_{J_{<\delta}} + \mathrm{HS}_{J_{\geq\delta}} - \mathrm{HS}_{\langle 0\rangle}$, and the fact that $\mathrm{HS}_{J_{<\delta}} - \mathrm{HS}_{\langle 0\rangle}$ is a polynomial in t, we can conclude that (Lemma 5.2) $N(t) = (1 - t^\delta \sum_{i=0}^n a_i(1-t)^i)/(1-t)^{n+1-d}$ where a_i is the number of monomial $m \in J_\delta$ such that i is the maximal integer ℓ with $x_\ell \mid m$. Let $s = 1-t$. Hence, $N(1-s) = (1-(1-s)^\delta \sum_{i=0}^n a_i s^i)/s^{n+1-d}$. Thus the degree of J which is $N(1)$ is the coefficient of s^{n+1-d} in $1-(1-s)^\delta \sum_{i=0}^n a_i s^i$ which is equal to $-\sum_{i=0}^{n-d+1}(-1)^i a_{n+1-d-i}\binom{\delta}{i}$. We summarize this result in the following theorem.

Theorem 6.3. *Let $J \subset R$ be a Borel type ideal. The degree of J is*

$$-\sum_{i=0}^{n-d+1}(-1)^i a_{n+1-d-i}\binom{\delta}{i}$$

where $d = \dim(J)$, $\delta = D_0 + \cdots + D_n - n$ with $x_0^{D_0} \cdots x_n^{D_n}$ the least common multiple of the minimal generators of J and a_i is the number of monomial $m \in J_\delta$ such that i is the maximal integer ℓ with $x_\ell \mid m$.

Remark 6.4. *Since the ideal $J + \langle x_n, \ldots, x_{n-d+1}\rangle$ is zero-dimensional, then $x_i^{D_i} \in J$ for $i = 0, \ldots, n-d$. Thus, $a_0 = 1$ and $a_i = \binom{i+\delta}{i} - a_{i-1} - \cdots - a_0$ for any $i < n-d+1$.*

Corollary 6.5. *Let $J = \langle m_1, \ldots, m_k\rangle$ be a Borel type ideal. The degree of J may be computed by $k(n+1-d)\binom{n-d+\delta}{n-d}$ operations.*

From this corollary, we conclude that the complexity of computing the degree of a Borel type ideal by this theorem is sharper than computing it using Hilbert series of the ideal.

Example 6.6. Computing the degree of a Borel type ideal. *Let us consider the ideal of Example 5.4. Its degree is equal to*

$$-(72\binom{12}{0} - 12\binom{12}{1} + \binom{12}{2}) = 6.$$

7. Conclusion

In this paper, we have presented a new algorithm which computes the Hilbert series of a Borel type ideal within a complexity polynomial in D^n where $n+1$ is the number of unknowns and D is the highest degree of a minimal generator of input polynomials. We have shown also that the satiety, Castelnuovo-Mumford regularity and degree of such an ideal may be computed within the above complexity.

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