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REFLECTION AND TRANSMISSION OF WAVES AT AN ELASTIC INTERFACE OF TWO HALF SPACES SUBJECT TO PURE SHEAR

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ABSTRACT. We study the effect of *pure shear* on the reflection and transmission of plane waves at the boundary between two half-spaces of incompressible isotropic elastic material. The half-spaces consist of the same material and are subjected to pure shear deformation with their principal axes aligned. The objective is to highlight the dependence of the amplitudes of the elastic waves on the finite pure shear deformation and thereby to provide a theoretical framework for the non-destructive evaluation at the shear interface.

When the first half-space corresponds to a certain class of constitutive laws and the second half-space *(not in this special class),* depending upon the angle of incidence, the material properties, and the magnitudes of deformations, it is shown that a homogeneous plane (SV) wave propagating in the plane of pure shear gives rise to a reflected wave (with angle of reflection equal to the angle of incidence) together with an interfacial wave in the same half-space, while in the other half-space there is a transmitted wave accompanied by an interfacial wave.

The dependence of the amplitudes of the reflected, transmitted, and interfacial waves on the angle of incidence and the states of deformation is illustrated graphically. The results described here provide a basis for the characterization of material properties and the finite homogeneous shear deformation.

1. Introduction

In [1] Hussain and Ogden have examined the effect of a finite simple shear deformation on the reflection of superimposed infinitesimal plane waves incident on the boundary of a half-space of incompressible isotropic elastic material. References to the literature concerned with reflection at the boundary of a finitely deformed half-space are contained in [2,3].

In the present paper the effect of *pure shear* on the reflection and *transmission* of plane (shear) waves at the boundary between two half-spaces which consist of the same material (but with *different* strain-energy functions) is considered. This problem of *mixed* strain-energy functions has *not* apparently been considered previously. The configuration is intended to describe the finite pure homogeneous shear deformation associated with the two half-spaces with a view to study theoretically the vibration and wave propagation characteristics of rubber like solids.

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The required equations and notations are summarized in Section 2. In Section 3 the propagation of plane harmonic waves is discussed with reference to the *slowness curves* appropriate for the two distinct classes of strain-energy functions.

The amplitudes of the reflected, transmitted and interfacial waves are calculated in Section 4 when a given homogeneous plane (shear) wave is incident on the boundary. *A combined case* of (distinct) strain-energy functions is discussed. In the paper by Dey and Addy [4] reflection and refraction of plane waves at an interface is discussed, which, as pointed out by Norris [5], contains fundamental errors.

For each angle of incidence a single reflected wave, with angle of reflection equal to the angle of incidence, is generated when a homogeneous plane (SV) wave is incident on the boundary from one half-space, and it is accompanied by an interfacial wave. In $x_2 > 0$ a transmitted wave and an interfacial wave are generated for *all* angles of incidence.

The theory in Section 4 is illustrated in Section 5 using graphical results to show the dependence of the amplitudes of the waves on the angle of incidence for representative values of the deformation parameters.

Finally, in Section 6 conclusions are given, describing the significance of the results obtained along with the research options for the extension of the analysis done in this paper.

2. Basic equations

We identify the undeformed configuration of the material, $\mathcal{B}_0$ say, and let a material particle in $\mathcal{B}_0$ be labelled by its three dimensional position vector $\mathbf{X}$. Let $\mathbf{x}$ be the position vector of the same particle in the deformed configuration, $\mathcal{B}$ say. We write the deformation of the material from $\mathcal{B}_0$ to $\mathcal{B}$, $\boldsymbol{\chi}$ say, as

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}), \quad \mathbf{X} \in \mathcal{B}_{\mathbf{0}}.$$

The deformation gradient tensor $\mathbf{A}$ is defined as

$$\mathbf{A} = \mathrm{Grad}\boldsymbol{\chi},$$

where Grad denotes the gradient with respect to $\mathbf{X}$, and is subject to the usual condition

$$\det \mathbf{A} > 0.$$

The polar decomposition theorem enables the second order tensor $\mathbf{A}$ to be written as

$$\mathbf{A} = \mathbf{VR},$$

where $\mathbf{R}$ is a proper orthogonal tensor and $\mathbf{V}$ is the symmetric and positive definite *left stretch tensor*.

Let $\mathbf{dX}$ is an arbitrary line element based at $\mathbf{X}$ in the reference configuration and $\mathbf{dx}$ is the corresponding line element at $\mathbf{x}$ in the deformed configuration. The *stretch* in the direction of $\mathbf{dX}$ at $\mathbf{X}$, is defined as the ratio of current to reference lengths of a line element and is given by

$$\frac{|\mathbf{dx}|}{|\mathbf{dX}|} = \lambda(\mathbf{M}),$$

where $\mathbf{M}$ is a unit vector along $\mathbf{dX}$.

The volume elements dV and dv in the reference and deformed configurations respectively are related by

$$dv = (det\mathbf{A})\, dV,$$

therefore for a volume preserving deformation we have

$$\det \mathbf{A} \equiv \lambda_1\lambda_2\lambda_3 = 1, \tag{1}$$

where $\lambda_i\,(>0)\,(i=1,2,3)$ are the eigenvalues (*principal stretches*), corresponding to the eigenvectors $\mathbf{v}_i\,(i=1,2,3)$, of the symmetric and positive definite tensor $\mathbf{V}$.

Let $\mathbf{S}$ denote the nominal stress tensor. Then, the equilibrium equation, in the absence of body forces, is

$$\mathrm{Div}\,\mathbf{S} = \mathbf{0},$$

where Div is the divergence operator in the reference configuration and $\mathbf{0} \in R^3$.

The measure of the energy stored per unit reference volume in the material as a result of deformation is called the elastic *stored energy function.* More commonly, the phrase *strain-energy function* is used to describe W (say) and this is the terminology we adopt in this paper.

For a (homogeneous) elastic material with strain-energy function $W = W(\mathbf{A})$ per unit volume, subject to the incompressibility constraint given by Eq. (1), we have

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{A}} - p\mathbf{A}^{-1},$$

where p is a Lagrange multiplier, which can be identified as a *hydrostatic pressure* associated with the incompressibility constraint. In general, it is a scalar function of time t, and in the literature is often introduced with the opposite sign.

An *isotropic elastic material* is an elastic material whose symmetry group contains the proper orthogonal group for at least one reference configuration. In such a reference configuration the mechanical response of the material exhibits no preferred direction, and it is this property that characterizes isotropy. If the material is isotropic, W depends symmetrically on $\lambda_1, \lambda_2, \lambda_3$ subject to Eq. (1) and we write $W(\lambda_1, \lambda_2, \lambda_3)$.

For the isotropic material, the principal Cauchy stresses are given by

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, \qquad i \in \{1,2,3\}.$$

For (plane strain) deformations confined to the $(1,2)$-plane, we may set $\lambda_3 = 1,$ so that Eq. (1) reduces to

$$\lambda_1\lambda_2 = 1.$$

Homogeneous *pure shear* deformation is defined by

$$\lambda_1 = \lambda \neq 1,\;\; \lambda_2 = \lambda^{-1},\, \lambda_3 = 1 \text{ with } \sigma_1 \neq 0,\, \sigma_2 = 0,$$

where a non-vanishing stress σ_3 is required to maintain $\lambda_3 = 1$. Superimposed on the deformation just described we consider incremental motions in the $(x_1,\, x_2)$-plane with displacement vector $\mathbf{v}$ having components

$$v_1(x_1, x_2, t), \quad v_2(x_1, x_2, t), \quad v_3 = 0.$$

The (linearized) incremental incompressibility condition $\operatorname{div}\mathbf{v} = 0$ enables v_1, v_2 to be expressed in terms of a scalar function, $\psi(x_1, x_2, t)$ say, so that

$$v_1 = \psi_{,2}, \qquad v_2 = -\psi_{,1}, \tag{2}$$

where $,i$ denotes $\partial/\partial x_i$, $i \in \{1, 2\}$.

The incremental nominal stress tensor is denoted by $\boldsymbol{\Sigma}$ when referred to the deformed configuration. Its components are given by

$$\Sigma_{ji} = \mathcal{A}_{0jilk} v_{k,l} + p v_{j,i} - \pi \delta_{ij}, \tag{3}$$

where π is the increment in p and $\mathcal{A}_{0jilk}$ are the components of the fourth-order tensor $\boldsymbol{\mathcal{A}}_\mathbf{0}$ of instantaneous elastic moduli (see, for example, Ogden [6]).

The components of $\boldsymbol{\mathcal{A}}_\mathbf{0}$ in terms of the derivatives of the strain-energy function W are given by

$$\begin{aligned} \mathcal{A}_{0iijj} &= \lambda_i \lambda_j W_{ij}, \\ \mathcal{A}_{0ijij} &= \frac{(\lambda_i W_i - \lambda_j W_j)\lambda_i^2}{(\lambda_i^2 - \lambda_j^2)} \qquad i \neq j, \quad \lambda_i \neq \lambda_j, \\ \mathcal{A}_{0ijij} &= \frac{1}{2}(\mathcal{A}_{0iiii} - \mathcal{A}_{0iijj} + \lambda_i W_i) \qquad i \neq j, \quad \lambda_i = \lambda_j, \\ \mathcal{A}_{0ijji} &= \mathcal{A}_{0jiij} = \mathcal{A}_{0ijij} - \lambda_i W_i \qquad i \neq j, \end{aligned} \tag{4}$$

where $W_i = \partial W/\partial\lambda_i$, $W_{ij} = \partial^2 W/\partial\lambda_i\partial\lambda_j$ and there is no summation over repeated indices. Here, the components $\mathcal{A}_{0jilk}$ are constants because the deformation under consideration is homogeneous.

The equation of motion is given by

$$\mathcal{A}_{0jilk} v_{k,jl} - \pi_{,i} = \rho \ddot{v}_i, \qquad i \in \{1, 2\}, \tag{5}$$

where ρ is the mass density of the material and there is summation from 1 to 2 over repeated indices. The equations of motion given by Eq. (5) yield, on restriction to the considered plane motion,

$$\begin{aligned} (\mathcal{A}_{01111} - \mathcal{A}_{01122} + p)v_{1,11} - \pi_{,1} + \mathcal{A}_{02121} v_{1,22} + (\mathcal{A}_{02121} - \sigma_2) v_{2,12} = \rho \ddot{v}_1, \\ (\mathcal{A}_{02222} - \mathcal{A}_{02211} + p)v_{2,22} - \pi_{,2} + \mathcal{A}_{01212} v_{2,11} + (\mathcal{A}_{02121} - \sigma_2) v_{1,12} = \rho \ddot{v}_2, \end{aligned} \tag{6}$$

where a superposed dot indicates the material time derivative.

Elimination of π from Eq. (6), and use of Eq. (2) yields an equation for ψ, namely

$$\alpha \psi_{,1111} + 2\beta \psi_{,1122} + \gamma \psi_{,2222} = \rho(\ddot{\psi}_{,11} + \ddot{\psi}_{,22}), \tag{7}$$

as given in [1], where the constants α, β, γ are defined by

$$\alpha = \mathcal{A}_{01212}, \quad \gamma = \mathcal{A}_{02121}, \quad 2\beta = \mathcal{A}_{01111} + \mathcal{A}_{02222} - 2\mathcal{A}_{01122} - 2\mathcal{A}_{01221}. \tag{8}$$

From Eq. (3), by using Eq. (2), the shear and normal components of the incremental nominal traction Σ_{21}, Σ_{22} on a plane $x_2 =$ constant are expressible in terms of ψ through

$$\begin{aligned} \Sigma_{21} &= \gamma \psi_{,22} - (\gamma - \sigma_2)\psi_{,11}, \\ -\Sigma_{22,1} &= (2\beta + \gamma - \sigma_2)\psi_{,112} + \gamma \psi_{,222} - \rho \ddot{\psi}_{,2}, \end{aligned} \tag{9}$$

in the latter of which the incremental hydrostatic pressure π has been eliminated by differentiating Σ_{22} with respect to x_1 and then using first equation in Eq. (6).

For any type of detail discussion, related to basic equations, the reader is requested to see Ogden [6].

3. Plane waves

We consider time-harmonic homogeneous plane waves of the form

$$\psi = A \exp[ik(x_1 \cos\theta + x_2 \sin\theta - ct)], \tag{10}$$

where A is a constant, c (> 0) the wave speed, k (> 0) the wave number and $(\cos\theta, \sin\theta)$ the direction cosines of the direction of propagation of the wave in the (x_1, x_2)-plane. Substitution of Eq. (10) into Eq. (7) gives

$$\alpha \cos^4\theta + 2\beta \sin^2\theta \cos^2\theta + \gamma \sin^4\theta = \rho c^2. \tag{11}$$

Equation (11) is a relationship between the wave speed and the propagation direction in the (x_1, x_2)-plane and is called the *propagation condition.* The material constants are taken to satisfy the strong ellipticity inequalities

$$\alpha > 0, \quad \gamma > 0, \quad \beta > -\sqrt{\alpha\gamma}, \tag{12}$$

and it is clear from Eq. (11) that $\rho c^2 > 0$ if and only if Eq. (12) hold.

Similarly, from Eq. (7), for an inhomogeneous plane wave of the form

$$\psi = \hat{A} \exp[ik'(x_1 - imx_2 - c't)], \tag{13}$$

we obtain

$$\alpha - 2\beta m^2 + \gamma m^4 = \rho(1 - m^2){c'}^2, \tag{14}$$

which relates the wave speed c' to the 'inhomogeneity factor' m. Note that the wave decays exponentially as $x_2 \to -\infty(+\infty)$ provided m has positive (negative) real part.

We now consider two half-spaces of the same incompressible isotropic elastic material. The half-spaces are subjected to pure shear deformation and then bonded along their common (plane) boundary in such a way that the principal directions of strain are aligned, one direction being normal to the interface.

Let λ_1, λ_2, λ_3 be the stretches associated with the half-spaces $x_2 < 0$, $x_2 > 0$, with strain energy function W and the material constants α, β, γ defined by Eq. (4) with Eq. (8).

We take the deformation to correspond to pure shear with $\lambda_3 = 1$ so that, with reference to the incompressibility condition (1), we introduce the notation λ such that

$$\lambda_1 = \lambda_2^{-1} = \lambda.$$

We consider two distinct cases corresponding to different strain-energy functions. For these either $2\beta = \alpha + \gamma$ or $2\beta \neq \alpha + \gamma$.

3.1. **Case A:** $2\beta = \alpha + \gamma$.

For this case equations Eq. (11) and Eq. (14) reduce to

$$\alpha \cos^2\theta + \gamma \sin^2\theta = \rho c^2 \tag{15}$$

and

$$(m^2 - 1)(\alpha - \gamma m^2 - \rho{c'}^2) = 0 \tag{16}$$

respectively.

In terms of the *slowness vector* (s_1, s_2) defined by

$$(s_1, s_2) = (\cos\theta, \sin\theta)/c$$

Eq. (15) becomes the *slowness* curve

$$\lambda^4 {s_1}^2 + {s_2}^2 = \overline{\rho}, \tag{17}$$

in the (s_1, s_2)-space, where $\overline{\rho}$ is defined by

$$\overline{\rho} = \rho/\gamma, \tag{18}$$

and $\alpha/\gamma = \lambda^4$ follows from Eq. (4) and Eq. (8).

By using the dimensionless notation $(\overline{s}_1, \overline{s}_2)$ defined by

$$(\overline{s}_1, \overline{s}_2) \equiv (s_1, s_2)/\sqrt{\overline{\rho}}, \tag{19}$$

we can write Eq. (17) as

$$\lambda^4 {\overline{s}_1}^2 + {\overline{s}_2}^2 = 1. \tag{20}$$

3.2. **Case B:** $2\beta \neq \alpha + \gamma$**.** In this case we take the strain-energy function to satisfy $\beta = \sqrt{\alpha\gamma}$ which was used by Hussain and Ogden in [1]. Then Eq. (11) takes the form

$$[\sqrt{\alpha}\cos^2\theta + \sqrt{\gamma}\sin^2\theta]^2 = \rho c^2 \tag{21}$$

and Eq. (14) becomes

$$(\sqrt{\alpha} - \sqrt{\gamma} m^2)^2 = \rho(1 - m^2){c'}^2. \tag{22}$$

The slowness curve corresponding to Eq. (21) is given by

$$[\lambda^2 \overline{s}_1^2 + \overline{s}_2^2]^2 = {\overline{s}_1}^2 + {\overline{s}_2}^2, \tag{23}$$

in dimensionless form with the notation given by Eq. (19) and $\overline{\rho}$ defined by Eq. (18). We now show graphically the dependence of the slowness curves on λ for both classes of strain-energy functions in $(\overline{s}_1, \overline{s}_2)$-space with reference to Eq. (20) and Eq. (23)(See Fig. 1-2).

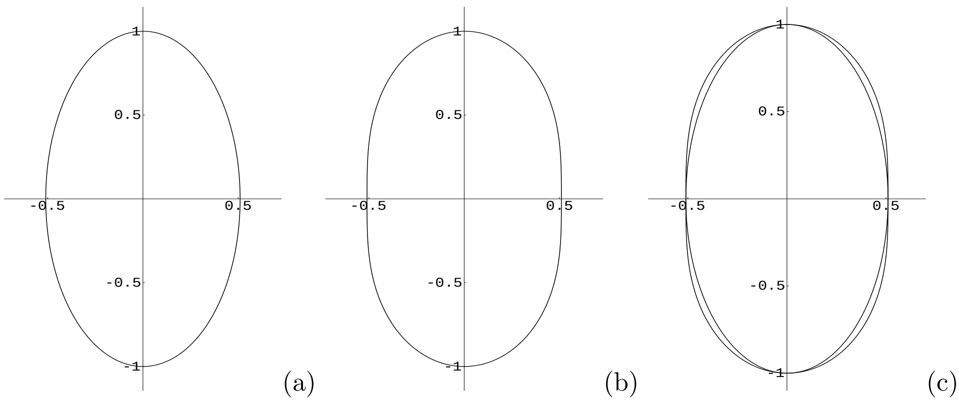

FIGURE 1. Slowness curves in $(\overline{s}_1, \overline{s}_2)$-space for $\lambda = 1.4$ with (a) $2\beta = \alpha + \gamma$, (b) $2\beta \neq \alpha + \gamma$, (c) the superposition of Figs. in (a) and (b).

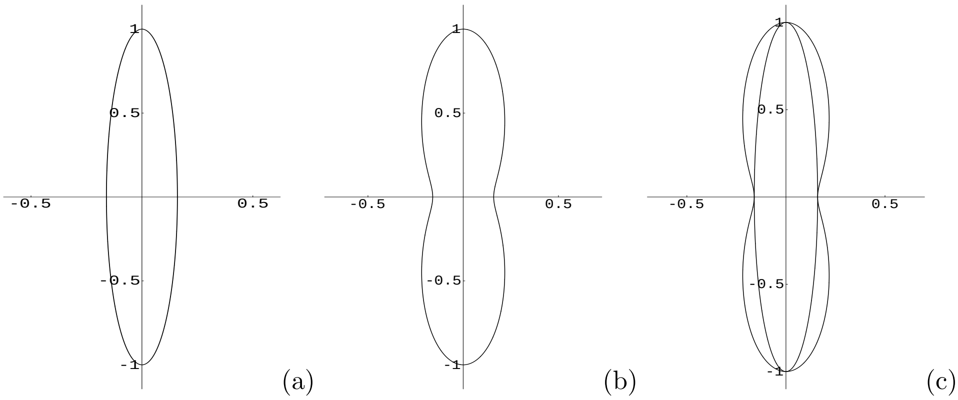

FIGURE 2. Slowness curves in $(\bar{s}_1, \bar{s}_2)$-space for $\lambda = 2.5$ with (a) $2\beta = \alpha + \gamma$, (b) $2\beta \neq \alpha + \gamma$, (c) the superposition of Figs. in (a) and (b).

4. Reflection and transmission at the interface

The boundary conditions corresponding to continuous displacement are $v_1 = v_1^*$, $v_2 = v_2^*$ on $x_2 = 0$, where v_1, v_2 are the displacement components in $x_2 < 0$ and v_1^*, v_2^* are those in $x_2 > 0$. From Eq. (2) these boundary conditions can be written in terms of the scalar functions ψ and ψ^* as

$$\psi_{,1} = \psi^*_{,1}, \ \psi_{,2} = \psi^*_{,2} \quad \text{on} \quad x_2 = 0, \tag{24}$$

where ψ^* is the counterpart of ψ for $x_2 > 0$.

The boundary conditions for continuous incremental traction on the interface are

$$\Sigma_{21} = \Sigma^*_{21}, \quad \Sigma_{22} = \Sigma^*_{22} \quad \text{on} \quad x_2 = 0, \tag{25}$$

where Σ_{21}, Σ_{22} are the traction components in $x_2 < 0$ and Σ^*_{21}, Σ^*_{22} are those in $x_2 > 0$.

From Eq. (9) the boundary conditions given by Eq. (25) take the forms

$$\begin{gathered} \psi_{,11} - \psi_{,22} = \psi^*_{,11} - \psi^*_{,22}, \\ (2\beta + \gamma)(\psi_{,112} - \psi^*_{,112}) + \gamma(\psi_{,222} - \psi^*_{,222}) - \rho(\ddot{\psi}_{,2} - \ddot{\psi}^*_{,2}) = 0, \end{gathered} \tag{26}$$

in terms of ψ and ψ^*, where, in order to obtain the second equation in Eq. (26), the second equation in Eq. (25) has been replaced by $\Sigma_{22,1} = \Sigma^*_{22,1}$ and use made of the second equation in Eq. (9) and its counterpart for $x_2 > 0$.

We now consider a wave incident on the boundary $x_2 = 0$ from the region $x_2 < 0$ with direction of propagation $(\cos\theta, \sin\theta)$ in the (x_1, x_2)- plane and speed c. Because of the symmetry of slowness curves with respect to the normal direction to the interface we henceforth, without loss of generality, restrict attention to values of θ in the interval $[0, \pi/2]$. We write the solution comprising the incident wave, a reflected wave (with angle of reflection equal to the angle of incidence) and an interfacial wave in $x_2 < 0$ as

$$\begin{aligned} \psi = {} & A\exp[ik(x_1\cos\theta + x_2\sin\theta - ct)] + AR\exp[ik(x_1\cos\theta - x_2\sin\theta - ct)] \\ & + AR'\exp[ik'(x_1 - imx_2 - c't)], \end{aligned} \tag{27}$$

where R is the reflection coefficient and R' measures the amplitude of the interfacial wave. The notations k', m, c' are as used in Eq. (13) and m has positive real part.

In the half-space $x_2 > 0$ we write the solution comprising a transmitted and an interfacial wave in the form

(28)
$$\psi^* = AR^* \exp[ik^*(x_1 \cos\theta^* + x_2 \sin\theta^* - c^* t)] + AR^{*'} \exp[ik^{*'}(x_1 + im^* x_2 - c^{*'} t)],$$

where R^* is the transmission coefficient and $R^{*'}$ is the analogue of R' for $x_2 > 0$. The transmitted wave has direction of propagation $(\cos\theta^*, \sin\theta^*)$, wave number k^* and speed c^*, while $k^{*'}$, m^*, $c^{*'}$ are the counterparts of k', m, c'. Note that the interfacial wave decays as $x_2 \to \infty$ provided m^* has positive real part.

According to the Snell's law we have

$$\cos\theta/c = 1/c^{'} = \cos\theta^*/c^* = 1/c^{*'}. \tag{29}$$

Eq. (29) states in particular, that the first components of the slowness vectors for each homogeneous plane wave interacting at the boundary $x_2 = 0$ are equal.

Thus, by reference to the slowness curves (superimposed) as exemplified in Fig. 1(c) and Fig. 2(c), the range of angles of incidence for which a transmitted wave exists can be identified. In Figs. 1(c) and 2(c), for example, if the *inner curve* corresponds to $x_2 < 0$ there is, for every angle of incidence (i.e. for every s_1 associated with the curve) a point on the outer curve (corresponding to $x_2 > 0$), and hence a transmitted wave.

We now examine here the case in which $2\beta = \alpha + \gamma\,(x_2 < 0)$, $2\beta \neq \alpha + \gamma\,(x_2 > 0)$. Analogous results, obtainable for $2\beta \neq \alpha + \gamma\,(x_2 < 0)$, $2\beta = \alpha + \gamma\,(x_2 > 0)$ will be discussed elsewhere.

4.1. $2\beta = \alpha + \gamma\,(x_2 < 0)$, $2\beta \neq \alpha + \gamma\,(x_2 > 0)$. In this case we see from Eq. (16) that $m = \pm 1$, which yields an interfacial wave in the half-space $x_2 < 0$ for $m = 1$. The zeros of the other quadratic factor in Eq. (16) correspond to $m = i\tan\theta$ and $m = -i\tan\theta$ which are associated, respectively, with the incident and reflected waves in $x_2 < 0$.

In $x_2 > 0$, from the counterpart of Eq. (22), after using $\alpha/\gamma = \lambda^4$ and Snell's law $\cos\theta^*/c^* = 1/c^{*'}$, we have

$$(m^{*2} + t^{*2})[m^{*2}(1 + t^{*2}) - t^{*2} + \lambda^2(\lambda^2 - 2)] = 0,$$

where $t^* = \tan\theta^*$.

The solution $m^* = it^*$ corresponds to a transmitted wave provided t^* is real and positive. The other relevant solution is

$$m^* = \pm\sqrt{1 - (\lambda^2 - 1)^2/(1 + t^{*2})} \tag{30}$$

with the plus sign when m^* is real. The nature of m^* in Eq. (30) depends on that of t^*, which is obtained by using the propagation condition given by Eq. (15) and the counterpart of Eq. (21) for the (transmitted) wave with direction of propagation $(\cos\theta^*, \sin\theta^*)$ and speed c^* together with Snell's law Eq. (29). This gives a quadratic for t^{*2}, which we write as

$$t^{*4} + t^{*2}(2\lambda^2 - \lambda^4 - t^2) - t^2 = 0, \tag{31}$$

and the notation $t = \tan\theta$ has been introduced. Note that t should be distinguished from the time variable t used earlier.

If t_1^{*2} and t_2^{*2} are the roots of Eq. (31) then we have

$$t_1^{*2} t_2^{*2} = -t^2,$$

which shows that there is one positive and one negative solution for t^{*2}, and hence one transmitted and one interfacial wave. See also Fig. 1(c) and Fig. 2(c).

When there is no refraction, i.e. a transmitted wave has the same direction of propagation as the incident wave ($\theta^* = \theta$). For this to be the case we must have $t^* = t$, and Eq. (31) gives $\lambda = 1$, which is *not* possible in case of pure shear deformation.

The coefficients R, R', R^* and $R^{*'}$ are determined by using the boundary conditions given by Eq. (24) and Eq. (26), with the second equation in Eq. (26) taking the form

$$(\lambda^4 + 2)\psi_{,112} - (2\lambda^2 + 1)\psi^*_{,112} + \psi_{,222} - \psi^*_{,222} - \overline{\rho}(\ddot{\psi}_{,2} - \ddot{\psi}^*_{,2}) = 0 \tag{32}$$

in this case, where $\overline{\rho}$ is given by Eq. (18). Substitution of ψ and ψ^* from Eq. (27)(with $m = 1$) and Eq. (28) in Eq. (24), the first equation in Eq. (26), and Eq. (32) leads to

$$\begin{aligned}
&1 + R + R' = R^* + R^{*'},\\
&t(1 - R) - iR' = t^* R^* + im^* R^{*'},\\
&(1 + R)(t^2 - 1) - 2R' = (t^{*2} - 1)R^* - (1 + m^{*2})R^{*'},\\
&t^{*2}\{2it(R - 1) + R'(t^2 - 1)\} + R^*(t^2 + t^{*2})it^* +\\
&R^{*'}\{t^{*4} + t^{*2}(m^{*2} - 1) - t^2\}m^* = 0.
\end{aligned} \tag{33}$$

In the latter equation use has been made of Eq. (31) in order to simplify the coefficients.

The solution of Eq. (33) may be written in the form

$$\begin{aligned}
R &= \frac{(t + i)(t^* - t)F(t)}{(t - i)(t^* + t)F(-t)},\\
R' &= \frac{2t(t - t^*)G'}{(t^* + i)(t - i)F(-t)},\\
R^* &= \frac{2t(t + i)G^*}{(i + t^*)(t^* - im^*)(t^* + t)F(-t)},\\
R^{*'} &= \frac{2t(i + t)(t^* - t)t^*}{i(m^* + it^*)F(-t)},
\end{aligned} \tag{34}$$

where $F(t)$, G', G^* are defined by

$$\begin{aligned}
F(t) &= t^2(t^* - i) - tt^* i(m^* + 1) + m^* t^{*2}(t^* + im^*),\\
G' &= m^* t^* (t^{*2} + im^* t^* + 1) - it^2,\\
G^* &= m^* t^{*4} + t^{*2}(t^2 + m^{*3} + m^{*2} + m^*) - m^* t^2,
\end{aligned}$$

and $F(-t)$ is obtained from $F(t)$ by replacing t by $-t$ without changing t^*. In these equations, for given t, m^* is obtained from Eq. (30) so as to have positive real part and t^* from Eq. (31). In Section 5 graphical results for the absolute values of R, R', R^* and $R^{*'}$ are given for illustration. All the figures have been produced using Mathematica [7].

5. Numerical Results

The slowness curves (superimposed) in Fig. 1(c) and Fig. 2(c) show that there is one reflected wave, one transmitted wave and two interfacial waves for each possible angle of incidence when $2\beta = \alpha + \gamma\,(x_2 < 0)$, $2\beta \neq \alpha + \gamma\,(x_2 > 0)$.

In Figs. 3-6, $|R|$, $|R'|$, $|R^*|$, $|R^{*'}|$ respectively are plotted, using Eq. (34), as functions of θ for a series of values of λ. Figs. (3-4) show, in particular, that as λ increases the maximum values of the reflected wave and the interfacial wave amplitudes (in $x_2 < 0$) $|R|$ and $|R'|$ increase.

In $x_2 > 0$, the character of the interfacial wave and the transmitted wave amplitudes $|R^*|$ and $|R^{*'}|$ is different (against different stretches).

For the grazing incidence ($\theta = 0$), from Eq. (31) we have

$$t^{*2} = \lambda^2(\lambda^2 - 2), \tag{35}$$

which shows that t^{*2} is positive when $\lambda > \sqrt{2}$ and negative for $\lambda < \sqrt{2}$. In Figs. 5(a-b) and Figs. 6(a-b) notice that $|R^*| \neq 0$ but $|R^{*'}| = 0$ when $\lambda < \sqrt{2}$ contrary to the results in Fig. 5(c-d) and Figs. 6(c-d) when $\lambda > \sqrt{2}$. In general the change in the amplitudes $|R^*|$ and $|R^{*'}|$ for $\lambda < \sqrt{2}$ and $\lambda > \sqrt{2}$ must be noted.

The graphical results show the general character of the effect of pure shear on the reflection and transmission of plane waves at the boundary of two half-spaces (corresponding to different strain-energy functions).

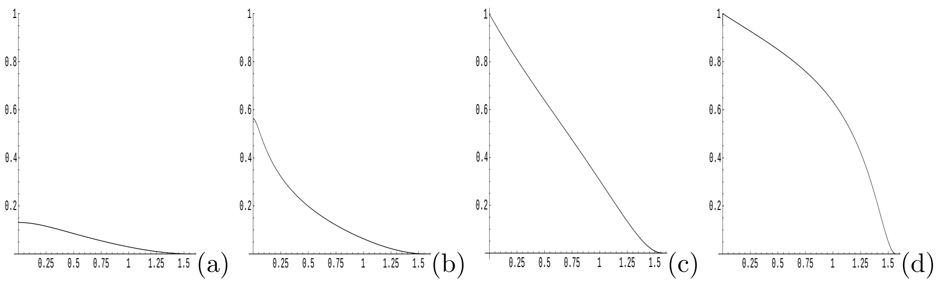

Figure 3. Plots of $|R|$ (reflected wave amplitude in $x_2 < 0$) against θ $(0 \leq \theta \leq \pi/2)$ with the following values of λ: (a) 0.6, (b) 1.4, (c) 1.9, (d) 2.8.

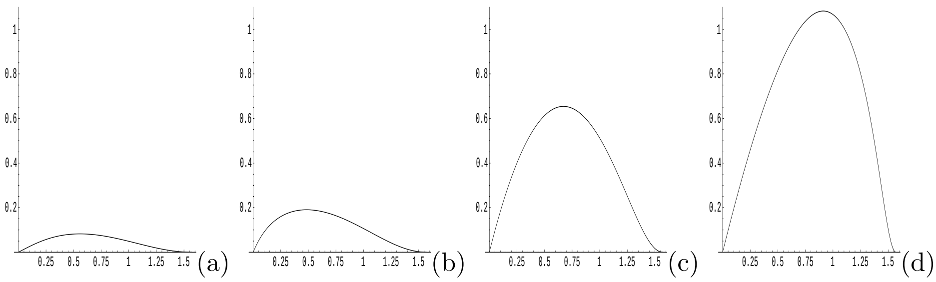

Figure 4. Plots of $|R'|$ (Interfacial wave amplitude in $x_2 < 0$) against θ ($0 \leq \theta \leq \pi/2$) with the following values of λ: (a) 0.6, (b) 1.4, (c) 1.9, (d) 2.8.

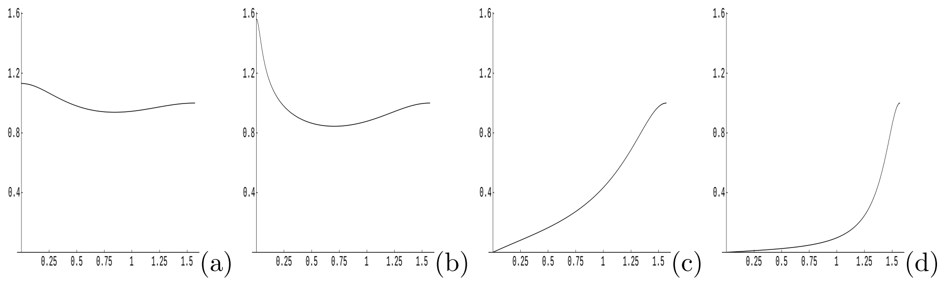

FIGURE 5. Plots of $|R^*|$ (Transmitted wave amplitude in $x_2 > 0$) against θ ($0 \leq \theta \leq \pi/2$) with the following values of λ: (a) 0.6, (b) 1.4, (c) 1.9, (d) 2.8.

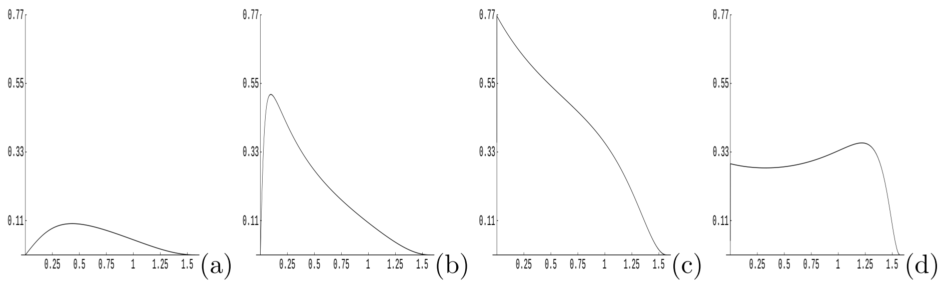

FIGURE 6. Plots of $|R^{*'}|$ (Interfacial wave amplitude in $x_2 > 0$) against θ ($0 \leq \theta \leq \pi/2$) with the following values of λ: (a) 0.6, (b) 1.4, (c) 1.9, (d) 2.8.

6. CONCLUSIONS

Since the angle of incidence θ is real, using first two equations in Eq. (34), the amplitudes of the reflected and interfacial waves are increasing smoothly (in $x_2 < 0$) by increasing the stretch as shown in Figs. (3-4).

As described in Section 5, at $\theta = 0$, $t^* = \tan\theta^*$ is real for $\lambda > \sqrt{2}$ from Eq. (35). Keeping this in view, the incident wave is not transmitted, as illustrated in Figs. 5(c-d). Do notice that interfacial wave is generated (in $x_2 > 0$) as shown in Figs. 6(c-d) for $\lambda > \sqrt{2}$.

Similarly θ^* is an imaginary angle from Eq. (35) for $\lambda < \sqrt{2}$. Therefore the behavior of $|R^*|$ ($|R^{*'}|$) at $\theta = 0$ for $\lambda < \sqrt{2}$ in Figs. 5(a-b) and Figs. 6(a-b) respectively, is similar to that of $|R^{*'}|$ ($|R^*|$) at $\theta = 0$ for $\lambda > \sqrt{2}$, as graphically shown in Figs. 6(c-d) and Figs. 5(c-d) respectively.

By comparing with the experimental data, the results described here, provide theoretical basis for characterization of material properties of rubber like solids i.e. stretches, strain-energy functions and elastic moduli. One can picture situations of practical interest where a solid is stretched and sheared: for instance, a rubber isolator under a bridge is subjected to vertical compression and then is sheared as a result of thermal extensions and contractions of the roadway.

The method and the results presented in the paper can be *extended* to the wave propagation problem by interchanging the strain-energy functions of two half-spaces i.e.

$$2\beta \neq \alpha + \gamma \,(x_2 < 0), \text{ and } 2\beta = \alpha + \gamma \,(x_2 > 0).$$

With reference to Fig. 1 problem would be similar i.e. there will be one reflected wave, one transmitted wave along with two interfacial waves *but* according to the Fig. 2, (in addition to Fig. 1 option) there might be *two* reflected waves (in $x_2 < 0$) with two interfacial waves (in $x_2 > 0$) with no transmitted wave for a certain range of the angle of incidence. Propagation of two interfacial waves (in the upper half-space), to best of the knowledge of author, *never* appeared in *linear* elasticity!

Another possible extension is to study the reflection and transmission of waves in unconstrained (*compressible)* elastic solids. See, for example [3], where reflection of plane waves from the boundary of a pre-stressed compressible elastic half-space is studied.

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