# $L_p$-NORM GENERALISED SYMMETRISED DIRICHLET DISTRIBUTIONS

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ABSTRACT. The paper deals with random vectors $\boldsymbol{X}$ possessing the stochastic representation $\boldsymbol{X} = R\boldsymbol{\mathcal{U}}$ where $R$ is a positive random radius and $\boldsymbol{\mathcal{U}}$ is a $L_p$-norm generalised symmetrised Dirichlet random vector independent of $R$. The Kotz Type I multivariate distribution appears prominently in the asymptotic results.

## 1. INTRODUCTION

Let $\boldsymbol{X}$ be a spherical random vector in $\mathbb{R}^k, k \geq 2$, i.e. the distribution function of $\boldsymbol{X}$ is invariant with respect to orthogonal transformations in $\mathbb{R}^k$. Define the associated random radius $R$ by the stochastic representation $R \stackrel{d}{=} (\sum_{i=1}^k X_i^2)^{1/2}$ ($\stackrel{d}{=}$ stands for equality in distribution). Cambanis et al. (1981) show in their pioneering paper that if $R > 0$ almost surely we have the stochastic representation

$$(1) \qquad \boldsymbol{X} \stackrel{d}{=} R\boldsymbol{\mathcal{U}},$$

with $\boldsymbol{\mathcal{U}}$ uniformly distributed on the unit sphere of $\mathbb{R}^k$ independent of the associated random radius $R$.

The main distributional properties of elliptical random vectors can be found in Kotz (1975), Cambanis et al. (1981), Anderson and Fang (1990), Fang et. al (1990), Fang and Zhang (1990), Szabłowski (1990), Berman (1992), Gupta and Varga (1993), Kano (1994), Kotz and Ostrovskii (1994) among many other sources.

When $\boldsymbol{\mathcal{U}}$ is uniformly distributed on the unit sphere of $\mathbb{R}^k$ the spherical distributions become quite tractable. On the other hand, due to these restrictions some important multivariate distributions such as Dirichlet distributions with unequal parameters do not belong to this class. Eliminating the assumption of the uniformity of the distribution function of $\boldsymbol{\mathcal{U}}$ allows studying more general distributions which share the simple stochastic representation (1).

Fang and Fang (1990) chose $\boldsymbol{\mathcal{U}}$ to have generalised symmetrised Dirichlet distribution (see below (2) for the definition) thus introducing generalised symmetrised Dirichlet random vectors in $\mathbb{R}^k$ with the stochastic representation (1). In the aforementioned paper several properties of this new class of random vectors are given. The well known in the theory and practice Dirichlet distribution was originally introduced by P.G.L. Dirichlet (a famous French-German mathematician in 1839).

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Another possible generalisation is to deal with the general $L_p$-norm ($p > 0$), but still retain the condition that $\mathcal{U}$ is uniformly distributed on the unit sphere of $\mathbb{R}^k$ with respect to the $L_p$-norm. This approach is suggested by Gupta and Song (1997) and Szabłowski (1998). Distributional properties of $L_p$-norm spherical random vectors derived in Gupta and Song (1997) and Szabłowski (1998) are, as expected, similar to the properties derived for $L_2$-norm spherical random vectors in Cambanis et al. (1981). Fang and Fang (1990), Gupta and Song (1997) and Szabłowski (1998) provide results that are shared by a wide class of multivariate distribution functions, and in particular by the class of spherical random vectors.

In this paper we shall consider a further generalisation (combining results in the aforementioned papers) by taking $\mathcal{U}$ in (1) to be a multivariate $L_p$-norm generalised symmetrised Dirichlet distribution (see below (2)), introducing thus via (1) $L_p$-norm generalised symmetrised Dirichlet random vectors.

We provide in this paper some basic distributional properties of $L_p$-norm generalised symmetrised Dirichlet (LpGSD) distributions. Furthermore, we obtain certain asymptotic results which are in line with the previous results for $L_p$-norm spherical random vectors. Conditional limiting theorems are derived in the last section. It is quite surprising that the standard Kotz Type I LpGSD distribution approximates a large subclass of LpGSD random vectors.

## 2. Notation and Preliminaries

For completeness we shall first present some notation and then review several known results about $L_p$-norm spherical random vectors and generalised symmetrised Dirichlet ones.

Let $I$ be a non-empty subset of $\{1, \ldots, k\}, k \geq 2$, and set $J := \{1, \ldots, k\} \setminus I$. For any vector $\boldsymbol{x} = (x_1, \ldots, x_k)^\top \in \mathbb{R}^k$ set $\boldsymbol{x}_I := (x_i, i \in I)^\top$, and write $\boldsymbol{x}_I^\top$ in place of $(\boldsymbol{x}_I)^\top$. Denote for two vectors in $\mathbb{R}^k, \boldsymbol{x}, \boldsymbol{y}$ the operations

$$\boldsymbol{x} + \boldsymbol{y} := (x_1 + y_1, \ldots, x_k + y_k),$$
$$\boldsymbol{x} > \boldsymbol{y}, \text{ if } x_i > y_i, \quad \forall i = 1, \ldots, k,$$
$$\boldsymbol{x} \geq \boldsymbol{y}, \text{ if } x_i \geq y_i, \quad \forall i = 1, \ldots, k,$$
$$\boldsymbol{x} \neq \boldsymbol{y}, \text{ if for some } i \leq k, x_i \neq y_i,$$
$$\boldsymbol{x} \not\leq \boldsymbol{y}, \text{ if for some } i \leq k, x_i > y_i,$$

and define

$$\boldsymbol{a}\boldsymbol{x} := (a_1 x_1, \ldots, a_k x_k)^\top, \quad c\boldsymbol{x} := (cx_1, \ldots, cx_k)^\top, \quad \boldsymbol{a} \in \mathbb{R}^k, c \in \mathbb{R},$$
$$\|\boldsymbol{x}_I\|_p := \left(\sum_{i \in I} |x_i|^p\right)^{1/p}, \quad p > 0, \quad (L_p\text{-norm}),$$
$$\mathcal{S}_p^{k-1} := \{\boldsymbol{x} \in \mathbb{R}^k : \|\boldsymbol{x}\|_p = 1\}, \quad (\text{unit sphere}).$$

Let moreover $\boldsymbol{0} := (0, \ldots, 0)^\top \in \mathbb{R}^k$, $\boldsymbol{1} := (1, \ldots, 1)^\top \in \mathbb{R}^k$. We shall be denoting by $Beta(a, b)$ and $Gamma(a, b)$ respectively, the distribution functions of a Beta or a Gamma random variables with parameters $a$ and $b$. If a random vector $\boldsymbol{Z}$ has the distribution function $Q$, this will be indicated by $\boldsymbol{Z} \sim Q$.

Throughout the paper $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_k), k \geq 2$ will denote a vector with positive components i.e. $\boldsymbol{\alpha} > 0$, $p$ be a fixed positive constant $(p > 0)$ and

$$\overline{\alpha} := \sum_{i=1}^{k} \alpha_i, \quad \overline{\alpha}_I := \sum_{i \in I} \alpha_i, \quad I \subset \{1, \ldots, k\}, |I| \geq 1.$$

The probability density function (p.d.f) of the Dirichlet distribution (see e.g. Kotz et al. (2000)) is given by

$$\frac{\Gamma(\overline{\alpha})}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \Big(1 - \sum_{i=1}^{k-1} u_i\Big)^{a_k - 1} \prod_{i=1}^{k-1} u_i^{\alpha_i - 1},$$

where $\sum_{i=1}^{k-1} u_i \leq 1, u_i > 0, i = 1, \ldots, k$. Denote by $(U_1, \ldots, U_{k-1})^\top$ a random vector with the above p.d.f and write $(U_1, \ldots, U_{k-1})^\top \sim \mathcal{D}(k, \boldsymbol{\alpha})$. The transformed random vector $(U_1^{1/p}, \ldots, U_{k-1}^{1/p})^\top$ has p.d.f

$$\frac{p^{k-1}\Gamma(\overline{\alpha})}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \Big(1 - \sum_{i=1}^{k-1} u_i^p\Big)^{a_k - 1} \prod_{i=1}^{k-1} u_i^{p\alpha_i - 1}, \quad u_i > 0, i = 1, \ldots, k : \sum_{i=1}^{k-1} |u_i|^p \leq 1.$$

Let $\mathcal{I}_1, \ldots, \mathcal{I}_k$ be independent random variables taking values $-1, 1$ with probability $1/2$. The random vector $(\mathcal{I}_1 U_1^{1/p}, , \ldots, \mathcal{I}_{k-1} U_{k-1}^{1/p})^\top$ represents a symmetrisation with power $p$ of $(U_1, \ldots, U_{k-1})^\top$. (For $p = 2$ it is referred in literature simply as symmetrisation). The p.d.f of the symmetrised random vector is thus

$$(2) \quad h(u_1, \ldots, u_{k-1}) \quad := \quad \frac{p^{k-1}\Gamma(\overline{\alpha})}{2^{k-1} \prod_{i=1}^{k} \Gamma(\alpha_i)} \Big(1 - \sum_{i=1}^{k-1} |u_i|^p\Big)^{a_k - 1} \prod_{i=1}^{k-1} |u_i|^{p\alpha_i - 1},$$

where $\sum_{i=1}^{k-1} |u_i|^p \leq 1$. (See (8) below for alternative derivation of this p.d.f).
Fang and Fang (1990) designate $\boldsymbol{\mathcal{U}} = (\mathcal{U}_1, \ldots, \mathcal{U}_k)^\top$ to have symmetrised Dirichlet distribution (with respect to $\boldsymbol{\alpha}$) provided $\|\boldsymbol{\mathcal{U}}\|_2 = 1$ and $\mathcal{U}_i = \mathcal{I}_i U_i^{1/2}, i = 1, \ldots, k-1$. We shall extend that definition as follows.

**Definition 1.** A random vector $\boldsymbol{\mathcal{U}}$ in $\mathbb{R}^k$ is said to have the $L_p$-norm symmetrised Dirichlet (LpSD) distribution with parameter $\boldsymbol{\alpha}$ if $\|\boldsymbol{\mathcal{U}}\|_p = 1$ almost surely and $(\mathcal{U}_1, \ldots, \mathcal{U}_{k-1})^\top$ has the p.d.f $h$ given by (2). (We shall denote $\boldsymbol{\mathcal{U}} \sim \mathcal{SD}(k, p, \boldsymbol{\alpha})$).

In some cases it may be more convenient to utilise unsymmetrised Dirichlet distributions. We denote

$$(3) \qquad\qquad \boldsymbol{\mathcal{U}} \quad \sim \quad \mathcal{D}(k, p, \boldsymbol{\alpha})$$

if $\boldsymbol{\mathcal{U}} \geq \boldsymbol{0}$ and $\|\boldsymbol{\mathcal{U}}\|_p = 1$ almost surely such that $(\mathcal{U}_1, \ldots, \mathcal{U}_{k-1})^\top$ has the density function $2^{k-1}h(u_1, \ldots, u_{k-1}), u_i > 0, i \leq k$, where $h$ is defined in (2).
Considering $\boldsymbol{\mathcal{U}}$ to be a $L_p$-norm symmetrised Dirichlet random vector with the stochastic representation (1) we arrive at the following definition.

**Definition 2.** A random vector $\boldsymbol{X}$ in $\mathbb{R}^k, k \geq 2$ is said to possess a $L_p$-norm generalised symmetrised Dirichlet distribution with parameter $\boldsymbol{\alpha}$ (denoted $\boldsymbol{X} \sim \mathcal{GSD}(k, p, \boldsymbol{\alpha}, F)$) if it possesses stochastic representation (1) where $R > 0$, almost surely with the distribution function $F$ independent of $\boldsymbol{\mathcal{U}}$, where $\boldsymbol{\mathcal{U}} \sim \mathcal{SD}(k, p, \boldsymbol{\alpha})$.

In the next section we shall derive some basic properties of the LpGSD random vectors and then proceed to Section 4 where we shall discuss dependence and asymptotic dependence of LpGSD distributions. Conditional limiting theorems motivated by previous results in Berman (1982,1983) and Berman (1992) are derived in Section 5. Section 5 focuses on the asymptotic tail behaviour in the case when the associated random radius is regularly varying. The proofs are relegated to Section 7. Further theoretical results are provided in the Appendix.

## 3. Main Distributional Properties

Using the derivations and definitions presented in Section 2 for a random vector $\boldsymbol{X} \sim \mathcal{GSD}(k, p, \boldsymbol{\alpha}, F)$ we have the following stochastic representation

$$(4) \qquad \boldsymbol{X} \stackrel{d}{=} R\boldsymbol{\mathcal{U}} \stackrel{d}{=} R(\mathcal{I}_1 U_1^{1/p}, \ldots, \mathcal{I}_{k-1} U_{k-1}^{1/p}, \mathcal{I}_k \mathcal{U}_k)^\top,$$

where $(U_1, \ldots, U_{k-1})^\top$ is a Dirichlet random vector with parameter $\boldsymbol{\alpha}$ and $\mathcal{U}_k > 0$ is such that the relation

$$\sum_{i=1}^{k-1} U_i + \mathcal{U}_k^p \;=\; 1$$

is valid almost surely.

The stochastic representation (3.1) shows that the role of parameter $p$ in the distributional properties of LpGSD random vectors is determined solely by the power transformation of the Dirichlet random vector $(U_1, \ldots, U_{k-1})^\top$. Although the distributional properties of Dirichlet random vectors are well-known (see e.g. Fang et al. (1990) or Kotz et al. (2000)), the main properties of LpGSD random vectors do not follow automatically. Further derivations are needed (as in Fang and Fang (1990)) to obtain the main distributional properties. Gupta and Song (1997) have shown that the $L_p$-norm spherical random vectors possess the same properties as the $L_2$-norm spherical (or simply spherical) random vectors. Here we shall show that the same is valid for LpGSD random vectors, utilising the techniques presented by Fang and Fang (1990).

First we shall observe that it is possible to arrive at the definition of the LpGSD random vectors via a single density generator (as it is presented in Fang and Fang (1990) for the $L_2$-norm).

Actually the definition of the density generator is unrelated to $p$ and is therefore similar to the one given in Fang and Fang (1990) presented below:

**Definition 3.** (Density generator) Let $g$ be a positive measurable function, and $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_k)^\top, k \geq 2$ be a given vector with positive components. If for some $\omega \in (0, \infty]$ the function $g$ satisfies

$$(5) \qquad \left(\frac{2}{p}\right)^k \frac{\prod_{i=1}^k \Gamma(\alpha_i)}{\Gamma(\overline{\alpha})} \int_0^\omega g(x) x^{\overline{\alpha}-1} \, dx = 1, \quad \overline{\alpha} := \sum_{i=1}^k \alpha_i,$$

we shall call $g$ to be a density generator with respect to $\boldsymbol{\alpha}$, denoting $g \sim \mathcal{G}(\boldsymbol{\alpha}, \omega)$. If the integral above is finite for any $\boldsymbol{\alpha}$ (with positive components) we shall refer to $g$ as the universal density generator.

The next result shows that a density generator uniquely defines the p.d.f of a LpGSD random vector.

**Theorem 1.** *Let $g \sim \mathcal{G}(\boldsymbol{\alpha}, \omega)$ with $\boldsymbol{\alpha} \in (0, \infty)^k, k \geq 2$, and $\omega \in (0, \infty]$, and $\boldsymbol{X}$ be a $k$-dimensional random vector with the density function $h$ defined by*

$$(6) \qquad h(\boldsymbol{x}) := g(\sum_{i=1}^{k} |x_i|^p) \prod_{i=1}^{k} |x_i|^{p\alpha_i - 1}, \quad \forall \boldsymbol{x} \in \mathbb{R}^k : 0 < \|\boldsymbol{x}\|_p < \omega,$$

*with $p > 0$. Then $\boldsymbol{X} \sim \mathcal{GSD}(k, p, \boldsymbol{\alpha}, F)$ where $F$ is a distribution function on $[0, \omega)$ with the p.d.f $f$*

$$(7) \qquad f(r) \;\; = \;\; 2\left(\frac{2}{p}\right)^{k-1} \frac{\prod_{i=1}^{k} \Gamma(\alpha_i)}{\Gamma(\overline{\alpha})} g(r^p) r^{p\overline{\alpha} - 1}, \quad \forall r \in (0, \omega).$$

*Conversely, if $\boldsymbol{X} \sim \mathcal{GSD}(k, p, \boldsymbol{\alpha}, F)$ with $F$ being a distribution function with the p.d.f $f$ then $\boldsymbol{X}$ possesses the density function $h$ defined in (6) with the density generator $g$ defined by the density $f$ in (7).*

In the case when $\boldsymbol{X}$ is defined in terms of a density generator $g \sim \mathcal{G}(\boldsymbol{\alpha}, \omega)$ we shall denote $\boldsymbol{X} \sim \mathcal{GSD}(k, p, \boldsymbol{\alpha}, g)$ suppressing the symbol $\omega$.

Several examples below should clarify the definitions and the theorem above.

**Example 1. [Symmetrised Dirichlet]** Let $\boldsymbol{\alpha} \in (0, \infty)^k, k \geq 2$ and $c, p$ be positive constants and specify $g(x) = c(1 - x)^{\alpha_k - 1}, \forall x \in (0, 1)$. Define the density function $h$ of a random vector $(\mathcal{U}_1, \ldots, \mathcal{U}_{k-1})^\top$ in $\mathbb{R}^{k-1}$ as in (6) by

$$h(\boldsymbol{x}) := c(1 - \sum_{i=1}^{k-1} |x_i|^p)^{\alpha_k - 1} \prod_{i=1}^{k-1} |x_i|^{p\alpha_i - 1}, \quad \boldsymbol{x} \in \mathbb{R}^{k-1} : 0 < \|\boldsymbol{x}\|_p^p < 1.$$

Utilising 5 we arrive at:

$$c^{-1} \;\; = \;\; \left(\frac{2}{p}\right)^{k-1} \frac{\prod_{i=1}^{k-1} \Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^{k-1} \alpha_i)} \int_0^1 x^{\sum_{i=1}^{k-1} \alpha_i - 1} (1 - x)^{\alpha_k - 1} \, dx$$

$$= \;\; \left(\frac{2}{p}\right)^{k-1} \frac{\prod_{i=1}^{k} \Gamma(\alpha_i)}{\Gamma(\overline{\alpha})}, \quad \overline{\alpha} := \sum_{i=1}^{k} \alpha_i.$$

Consequently we have for any $\boldsymbol{x} \in \mathbb{R}^{k-1}$ such that $\|\boldsymbol{x}\|_p < 1$

$$(8) \quad h(x_1, \ldots, x_{k-1}) \;\; = \;\; \left(\frac{p}{2}\right)^{k-1} \frac{\Gamma(\overline{\alpha})}{\prod_{i=1}^{k} \Gamma(\alpha_i)} (1 - \sum_{i=1}^{k-1} |x_i|^p)^{\alpha_k - 1} \prod_{i=1}^{k-1} |x_i|^{p\alpha_i - 1}.$$

**Example 2. [Kotz Type I]** Let the density generator $g$ be of the form

$$(9) \qquad g(x) \;\; = \;\; cx^N \exp(-rx^s), \quad x > 0, c > 0, N \in \mathbb{R}, r > 0, s > 0.$$

For a given $\boldsymbol{\alpha}$ and restricting $N > -\overline{\alpha}$ we obtain using (5) that the constant $c$ is determined by

$$c\left(\frac{2}{p}\right)^k \frac{\prod_{i=1}^{k} \Gamma(\alpha_i)}{\Gamma(\overline{\alpha})} \int_0^\infty x^N \exp(-rx^s) x^{\overline{\alpha} - 1} \, dx \;\; = \;\; 1.$$

Observing that

$$(10) \qquad \int_0^\infty x^N \exp(-rx^s) x^{\overline{\alpha} - 1} \, dx \;\; = \;\; \frac{\Gamma((N + \overline{\alpha})/s)}{sr^{(N+\overline{\alpha})/s}}$$

we arrive at:

$$(11) \qquad c \quad := \quad \left(\frac{p}{2}\right)^k \frac{sr^{(N+\overline{\alpha})/s}}{\Gamma((N+\overline{\alpha})/s)} \frac{\Gamma(\overline{\alpha})}{\prod_{i=1}^k \Gamma(\alpha_i)}.$$

Here the density generator $g$ given by (9) is a universal one.

We say that $\boldsymbol{X}$ in $\mathbb{R}^k$ is a Kotz Type I LpGSD random vector if its density function $h$ is given for any $\boldsymbol{x} \in \mathbb{R}^k$ by

$$(12) \quad h(\boldsymbol{x}) := \left(\frac{p}{2}\right)^k \frac{sr^{(N+\overline{\alpha})/s}}{\Gamma((N+\overline{\alpha})/s)} \frac{\Gamma(\overline{\alpha})}{\prod_{i=1}^k \Gamma(\alpha_i)} \|\boldsymbol{x}\|_p^{pN} \exp(-r\|\boldsymbol{x}\|_p^{ps}) \prod_{i=1}^k |x_i|^{p\alpha_i - 1},$$

involving the norm of $\boldsymbol{x}$, an exponential function and a product of the components of $\boldsymbol{x}$ (compare with (8)). In the standardised case $N = 0$ and $r = s = 1$ the random vector $\boldsymbol{X}$ possesses independent components such that

$$(13) \qquad |X_i|^p \sim Gamma(\alpha_i, 1/p), \quad \forall i = 1, \dots, k.$$

We shall denote by $\boldsymbol{\mathcal{K}}_{\boldsymbol{\alpha}, p}$ the distribution function of $\boldsymbol{X}$ when $N + 1 = r = s = 1$.

**Example 3. [Kotz Type II]** Let the density generator $g$ be of the form

$$(14) \qquad g(x) \quad = \quad cx^N \exp(-rx^s), \quad x > 0, c > 0, N < 0, r > 0, s < 0.$$

Here the values of $N$ and $s$ are negative. Analogously to the previous example, for a given $\boldsymbol{\alpha}$, the constant $c$ is obtained from (5)

$$c \left(\frac{2}{p}\right)^k \frac{\prod_{i=1}^k \Gamma(\alpha_i)}{\Gamma(\overline{\alpha})} \int_0^\infty x^N \exp(-rx^s) x^{\overline{\alpha} - 1} \, dx = 1.$$

Choosing $N < -\overline{\alpha}$ we obtain

$$c := \left(\frac{p}{2}\right)^k \frac{(-s)r^{(N+\overline{\alpha})/s}}{\Gamma((N+\overline{\alpha})/s)} \frac{\Gamma(\overline{\alpha})}{\prod_{i=1}^k \Gamma(\alpha_i)} > 0.$$

Here $g$ is also a universal density generator.

We define $\boldsymbol{X}$ in $\mathbb{R}^k$ to be a Kotz Type II LpGSD random vector provided its p.d.f $h$ is given for $N < -\overline{\alpha}$ and $\boldsymbol{x} \in \mathbb{R}^k$ by

$$h(\boldsymbol{x}) := \left(\frac{p}{2}\right)^k \frac{|s|r^{(N+\overline{\alpha})/s}}{\Gamma((N+\overline{\alpha})/s)} \frac{\Gamma(\overline{\alpha})}{\prod_{i=1}^k \Gamma(\alpha_i)} \|\boldsymbol{x}\|_p^{pN} \exp(-r\|\boldsymbol{x}\|_p^{ps}) \prod_{i=1}^k |x_i|^{p\alpha_i - 1}.$$

The $L_p$-norm Kotz Type II spherical random vectors are considered in Hashorva (2006d). The original definition of these random vectors for the $L_2$-norm case is due to Kotz (1975).

**Example 4. [Kotz Type III]** Let $\boldsymbol{X} = R\boldsymbol{U}$ with $R$ a positive random radius independent of the $k$-dimensional random vector $\boldsymbol{U}$ which is such that $\|\boldsymbol{U}\|_p = 1$ almost surely. We refer to $\boldsymbol{X}$ as a Kotz Type III random vector if the associated random radius $R > 0$ has asymptotic tail behaviour ($u \to \infty$)

$$(15) \quad \boldsymbol{P}\{R > u\} \quad = \quad (1 + o(1))Ku^N \exp(-ru^\delta) \quad K > 0, \delta \in \mathbb{R}, N \in \mathbb{R}, r > 0.$$

For $\delta \leq 0$ we assume that $N < 0$. If $\boldsymbol{U}$ is a LpGSD random vector then $\boldsymbol{X}$ is a LpGSD random vector.

Both Kotz Type I and Type II LpGSD random vectors belong to the larger class of the Kotz Type III random vectors.

**Example 5.** [**Pearson Type VII**] The density generator is $g(x) = c(1 + t/s)^{-N}$ with $c, s$ positive constants. Assuming $N > \overline{\alpha}$ we obtain the density function $h$ of a $k$-dimensional LpGSD Pearson Type VII distribution

$$h(\boldsymbol{x}) = \left(\frac{p}{2}\right)^k s^{-\overline{\alpha}} \frac{\Gamma(N)}{\Gamma(N - \overline{\alpha}) \prod_{i=1}^k \Gamma(\alpha_i)} (1 + \sum_{i=1}^k |x_i|^p/s)^N \prod_{i=1}^k |x_i|^{p\alpha_i - 1},$$

for all $\boldsymbol{x} \in \mathbb{R}^k$.

**Example 6.** [**Kummer-Beta**] Let $g$ be a density generator of a Kummer-Beta LpGSD distribution given by

$$g(x) = cx^{\delta-1}(1 - x)^{\gamma-1}\exp(-\lambda x), \quad 0 < x < 1, \delta > 0, \lambda \geq 0, \gamma > 0.$$

The normalising constant $c$ for given positive constants $\alpha_i, i \leq k$ such that $\overline{\alpha} > 1 - \delta$ is specifically determined via the relations:

$$
\begin{aligned}
c^{-1} &= \left(\frac{2}{p}\right)^k \frac{\prod_{i=1}^k \Gamma(\alpha_i)}{\Gamma(\overline{\alpha})} \int_0^1 x^{\overline{\alpha}-1}\exp(-\lambda x)x^{\delta-1}(1-x)^{\gamma-1}\,dx \\
&= \left(\frac{2}{p}\right)^k \frac{\prod_{i=1}^k \Gamma(\alpha_i)}{\Gamma(\overline{\alpha})} \int_0^1 x^{\overline{\alpha}+\delta-2}(1-x)^{\gamma-1}\exp(-\lambda x)\,dx \\
&= \left(\frac{2}{p}\right)^k \frac{\prod_{i=1}^k \Gamma(\alpha_i)}{\Gamma(\overline{\alpha})} \, {}_1F_1(\overline{\alpha}+\delta-1;\overline{\alpha}+\delta+\gamma-1;-\lambda)\frac{\Gamma(\overline{\alpha}+\delta-1)\Gamma(\gamma)}{\Gamma(\overline{\alpha}+\delta+\gamma-1)},
\end{aligned}
$$

where ${}_1F_1$ is the confluent hypergeometric function of the first kind (also known as Kummer's function of the first kind). ${}_1F_1$ has a hypergeometric series expansion given by

$${}_1F_1(a, b, x) = 1 + \frac{a}{b}x + \frac{a(a+1)}{b(b+1)}\frac{x^2}{2!} + \cdots = \sum_{k=0}^\infty \frac{(a)_k}{(b)_k}\frac{x^k}{k!},$$

where $(a)_k, (b)_k$ are the Pochhammer symbols.

**Example 7.** [**Kummer-Gamma**] The density generator $g$ of a Kummer-Gamma LpGSD distribution is specified as

$$g(x) = cx^{\delta-1}(1 + x)^{\gamma-1}\exp(-\lambda x), \quad x > 0, \delta > 0, \lambda > 0, \gamma > 0,$$

with

$$
\begin{aligned}
c^{-1} &= \left(\frac{2}{p}\right)^k \frac{\prod_{i=1}^k \Gamma(\alpha_i)}{\Gamma(\overline{\alpha})} \int_0^\infty x^{\overline{\alpha}-1}\exp(-\lambda x)x^{\delta-1}(1+x)^{\gamma-1}\,dx \\
&= \left(\frac{2}{p}\right)^k \frac{\prod_{i=1}^k \Gamma(\alpha_i)}{\Gamma(\overline{\alpha})} \Psi(\overline{\alpha}+\delta-1;\overline{\alpha}+\delta-\gamma-1;\lambda)\Gamma(\overline{\alpha}+\delta-1),
\end{aligned}
$$

where $\Psi$ is the confluent hypergeometric function of the second kind. It is also known as the Kummer's function of the second kind, Tricomi function, or Gordon function.

See Kotz and Ng (1995) for some basic properties of the Kummer-Beta and Kummer-Gamma distributions. We note in passing that a Kummer-Gamma LpGSD random vector belongs to the class of Kotz Type III LpGSD random vectors defined in Example 4.

In addition to the fulfillment of the stochastic representation (1) the most distinguishing property of LpGSD distributions is the so-called amalgamation property, initially presented in Cambanis at al. (1981) for elliptical random vectors, and in

Fang and Fang (1990), Gupta and Song (1997) for generalised symmetrised Dirichlet and $L_p$-norm spherical random vectors, respectively.

**Theorem 2** (Amalgamation property). *Let $I_1, \ldots, I_m, m \geq 2$ be a partition of $\{1, \ldots, k\}, k \geq 1$ and $\boldsymbol{X} \sim \mathcal{GSD}(k, p, \boldsymbol{\alpha}, F)$ be a $k$-dimensional random vector as in Theorem 1. Then for any $j = 1, \ldots, m$ we have the stochastic representation*

$$(16) \qquad\qquad \boldsymbol{X}_{I_j} \;\overset{d}{=}\; RW_j \boldsymbol{Z}_{I_j},$$

*where the variables $R, \boldsymbol{W}_m := (W_1, \ldots, W_m)^\top, \boldsymbol{Z}_{I_1}, \ldots, \boldsymbol{Z}_{I_m}$ are pairwise independent random vectors with the random variable $R > 0$, and*

$$(17) R \sim F, \quad \boldsymbol{W}_m \sim \mathcal{D}(m, p, \boldsymbol{a}_m), \quad \boldsymbol{Z}_{I_j} \sim \mathcal{SD}(k_j, p, \boldsymbol{\alpha}_{I_j}), \quad j = 1, \ldots, m,$$

*with $\boldsymbol{a}_m := (\sum_{i \in I_1} \alpha_i, \ldots, \sum_{i \in I_m} \alpha_i)^\top, k_j = |I_j| \geq 1$ and $\mathcal{SD}(k_j, p, \boldsymbol{\alpha}_{I_j}),$ $\mathcal{D}(m, p, \boldsymbol{a})$ as in Definition 2.1 and 2.2, respectively.*

For any non-empty index set $I$ and $p > 0$ we define the associated random radius $R_{I,p}$ of $\boldsymbol{X}$ by

$$R_{I,p} := \Big( \sum_{i \in I} |X_i|^p \Big)^{1/p} = \|\boldsymbol{X}_I\|_p > 0, \quad p > 0.$$

In the case $I = \{1, \ldots, k\}$ we shall simply write $R$ instead of $R_{I,p}$.

**Corollary 3.** *Let $\boldsymbol{X}$ be defined as in Theorem 2, and $R_{I,p}$ be the associated random radius of $\boldsymbol{X}$ with respect to the non-empty index set $I$ with $m$ elements. Then the stochastic representation*

$$(18) \qquad\qquad \boldsymbol{X}_I \;\overset{d}{=}\; R_{I,p} \boldsymbol{\mathcal{V}}_I$$

*is valid with $R_{I,p}$ independent of $\boldsymbol{\mathcal{V}}_I$ where $\boldsymbol{\mathcal{V}}_I \sim \mathcal{SD}(m, p, \boldsymbol{\alpha}_I)$. Furthermore, if $m < k$*

$$(19) \qquad\qquad R_{I,p}^p \;\overset{d}{=}\; R^p W$$

*holds where $W > 0$ is distributed as $W \sim Beta(\overline{\alpha}_I, \overline{\alpha} - \overline{\alpha}_I)$, with $W, R$ independent.*

**Corollary 4.** *Let $\boldsymbol{X} \sim \mathcal{GSD}(k, p, \boldsymbol{\alpha}, F)$ be a LpGSD random vector in $\mathbb{R}^k$. Then we have the stochastic representation*

$$(20) \qquad X_j \overset{d}{=} R\mathcal{I}_j \Big[ |\cos(\Theta_j)| \prod_{i=1}^{j-1} \sin(\Theta_i) \Big]^{2/p}, \quad 1 \leq j \leq k-1,$$

$$(21) \qquad\qquad X_k \overset{d}{=} R\mathcal{I}_k \Big[ |\sin(\Theta_{k-1})| \prod_{i=1}^{k-2} \sin(\Theta_i) \Big]^{2/p},$$

*where $\mathcal{I}_j = \mathrm{sign}(\cos(\Theta_j)), 1 \leq j \leq k-1, \mathcal{I}_k = \mathrm{sign}(\sin(\Theta_{k-1}))$ are independent random variables, being further independent of the random angles $\Theta_i, 1 \leq i \leq k-1$ which have the density functions*

$$q_i(\theta) := \frac{\Gamma(\overline{\alpha}_J)}{\Gamma(\overline{\alpha}_J - \alpha_i)\Gamma(\alpha_i)} |\sin(\theta)|^{2\overline{\alpha}_J - 1} |\cos(\theta)|^{2\alpha_i - 1}, \quad 0 \leq \theta \leq \pi, 1 \leq i \leq k-2,$$

*where $J := \overline{\alpha} - \sum_{j=1}^{i} \alpha_i$, and*

$$q_{k-1}(\theta) := \frac{1}{2} \frac{\Gamma(\alpha_{k-1} + \alpha_k)}{\Gamma(\alpha_{k-1})\Gamma(\alpha_k)} |\sin(\theta)|^{2\alpha_k - 1} |\cos(\theta)|^{2\alpha_{k-1} - 1}, \quad 0 \leq \theta \leq 2\pi.$$

*Furthermore, $R, \Theta_i, 1 \leq i \leq k-1$ are independent random variables and $R \sim F$. Conversely, if (20),(21) holds with $R, \Theta_i, 1 \leq i \leq k$ independent random variables where $R \sim F$ is a positive random radius and the random angle $\Theta_i$ has the density function $q_i$ defined above, then $\boldsymbol{X} \sim \mathcal{GSD}(k, p, \boldsymbol{\alpha}, F)$.*

**Remark 1.** a) In view of Corollary 3, any subvector $\boldsymbol{X}_I, I \subset \{1, \ldots, k\}$ of a $k$-dimensional LpGSD random vector $\boldsymbol{X}$ has a LpGSD distribution function. Moreover $\boldsymbol{X}_I$ possesses a density function. This property was derived for elliptical distributions in Cambanis et al. (1981). Explicitly, let $\boldsymbol{X}_I$ be as defined in Corollary 3, then if follows from (19) that the associated random radius $R_{I,p}$ possesses the density function $f$ given for any $u \in (0, \omega)$ by

$$(22) \quad f(u) \;=\; pu^{p\overline{\alpha}_I - 1} \frac{\Gamma(\overline{\alpha})}{\Gamma(\overline{\alpha}_I)\Gamma(\overline{\alpha} - \overline{\alpha}_I)} \int_u^\omega (r^p - u^p)^{\overline{\alpha} - \overline{\alpha}_I - 1} r^{-p(\overline{\alpha}-1)} \, dF(r).$$

Here $\omega$ is the upper endpoint of the distribution function $F$ of $R$.

In the special case $\alpha_i = 1/p, i = 1, \ldots, k$, the expression for the p.d.f simplifies to (see e.g. Gupta and Song (1997))

$$(23) f(u) \;=\; pu^{m-1} \frac{\Gamma(k/p)}{\Gamma(m/p)\Gamma((k-m)/p)} \int_u^\infty (r^p - u^p)^{(k-m)/p - 1} r^{-k+p} \, dF(r).$$

b) If $\alpha_i = i/p, 1 \leq i \leq k$, then Corollary 4 reduces to Theorem 2 in Szabłowski (1998). The case of $L_2$-norm spherical random vectors is presented in Theorem 2.11 of Fang et al. (1990).

Next we derive the conditional distribution $\boldsymbol{X}_I | \boldsymbol{X}_J = \boldsymbol{x}_J, \boldsymbol{x} \in \mathbb{R}^k$ where $I, J$ are two non-empty disjoint index sets of $\{1, \ldots, k\}$. It follows that the conditional distribution is determined in terms of the norm $\|\boldsymbol{x}_J\|_p := (\sum_{j \in J} |x_j|^p)^{1/p}$.

We again emphasise that the results obtained in this section are similar to those for a much narrower classes of spherical and elliptical random vectors. Namely, the asymptotic results remain valid when in the basic stochastic representation (1), the $L_2$-norm uniformly distributed random vector $\boldsymbol{\mathcal{U}}$ is replaced by a $L_p$-norm generalised symmetrised Dirichlet random vector.

**Theorem 5.** *Let $\boldsymbol{X} \sim \mathcal{GSD}(k, p, \boldsymbol{\alpha}, F)$, with $p > 0, \boldsymbol{\alpha} \in (0, \infty)^k, k \geq 1$, and let $I, J$ be partitions of $\{1, \ldots, k\}$. Then for any $\boldsymbol{x} \in \mathbb{R}^k$ with $F(\|\boldsymbol{x}_J\|_p) \in (0, 1)$ we have*

$$(24) \qquad \boldsymbol{X}_I | \boldsymbol{X}_J = \boldsymbol{x}_J \;\stackrel{d}{=}\; R_{\|\boldsymbol{x}_J\|_p} \boldsymbol{\mathcal{V}}_I, \quad \boldsymbol{\mathcal{V}}_I \sim \mathcal{SD}(m, p, \boldsymbol{\alpha}_I).$$

*Moreover $\boldsymbol{\mathcal{V}}_I$ is independent of $R_{\|\boldsymbol{x}_J\|_p} > 0$ with the distribution function $G$ given by*

$$(25) \; G(x) := 1 - \frac{\int_{(\|\boldsymbol{x}_J\|_p^p + x^p)^{1/p}}^\omega (r^p - \|\boldsymbol{x}_J\|_p^p)^{\overline{\alpha}_I - 1} r^{-p\overline{\alpha}+p} \, dF(r)}{\int_{\|\boldsymbol{x}_J\|_p}^\omega (r^p - \|\boldsymbol{x}_J\|_p^p)^{\overline{\alpha}_I - 1} r^{-p\overline{\alpha}+p} \, dF(r)}, \quad \forall x > 0,$$

*where $\omega \in (0, \infty]$ is the upper endpoint of the distribution function $F$.*

## 4. DEPENDENCE AND ASYMPTOTIC DEPENDENCE

A simple example of elliptical random vectors is $\boldsymbol{X} \sim N(\boldsymbol{\mu}, \Sigma)$ a Gaussian random vector in $\mathbb{R}^k, k \geq 2$, with the covariance matrix $\Sigma$ and mean vector $\boldsymbol{\mu}$. It is well-known (see e.g. Fang et al. (1990)) that the independence of the components of $\boldsymbol{X}$ is equivalent to the assumption that $\Sigma$ is the identity matrix. It is also

well-known (see e.g. Cambanis et al. (1981), Fang et al. (1990)) that a spherical random vector has independent components iff its components are Gaussian. Fang and Fang (1990) provide several conditions which imply the independence of the components of $L_2$-norm generalised symmetrised Dirichlet random vectors. In the next theorem we shall show that similar conditions are valid for a more general case of LpGSD random vectors.

Also it follows from the Theorem 6 that independence of components holds only in the case of Kotz Type I LpGSD distribution with parameters $N + 1 = s = 1$ and $r > 0$.

**Theorem 6.** *Let $\boldsymbol{X}$ be a $L_p$-norm generalised symmetrised Dirichlet random vector in $\mathbb{R}^k, k \geq 2$, with the density generator $g \sim \mathcal{G}(\boldsymbol{\alpha}, \omega)$. The following statements are equivalent:*

*(1) $\boldsymbol{X}$ possesses independent components.*

*(2) For any $I \subset \{1, \ldots, k\}$ the random vector $\boldsymbol{X}_I$ has Kotz Type I LpGSD distribution with parameters $N = 0, s = 1$ and $r > 0$.*

*(3) There exist $I, J$ disjoint index sets such that $\boldsymbol{X}_I$ is independent of $\boldsymbol{X}_J$.*

*(4) There exist $I, J$ disjoint index sets with $I \cup J \subset \{1, \ldots, k\}$ such that $\boldsymbol{X}_I | \boldsymbol{X}_J$ is independent of $\boldsymbol{X}_J$.*

*(5) For any $I \subset \{1, \ldots, k\}$ we have $R_{I,p}^p \sim \Gamma(\overline{\alpha}_I, r)$ with $\overline{\alpha}_I = \sum_{i \in I} \alpha_i$ and $r$ is a positive constant.*

*(6) There exist $I, J$ disjoint index sets with $I \cup J \subset \{1, \ldots, k\}$ (provided $\overline{\alpha}_I \neq \overline{\alpha}_J$) such that the density generators of $\boldsymbol{X}_I$ and $\boldsymbol{X}_J$ differ only up to a positive constant.*

The assumption of the above theorem that $\boldsymbol{X}$ possesses a density function is somewhat restrictive. In view of Theorem 2 any subvector $\boldsymbol{X}_I$, with $1 \leq |I| < k$ possesses a density function even when $\boldsymbol{X}$ does not posses one. Thus if $k \geq 2$, the assumption that $\boldsymbol{X}$ has a density generator is not needed. Several statements given above could then be easily reformulated.

Next, we shall discuss the asymptotic dependence of LpGSD random vectors. Let $\boldsymbol{X}$ be as in Theorem 6 with the associated random radius $R \sim F$. A meaningful parameter for the asymptotic dependence between the components $X_i, X_j, 1 \leq i < j < k$, is the limit (provided it exists)

$$\tau(X_i, X_j) := \lim_{t \uparrow \omega} \frac{\boldsymbol{P}\{X_i > t, X_j > t\}}{\boldsymbol{P}\{X_i > t\} + \boldsymbol{P}\{X_j > t\}},$$

where $\omega := \sup\{x : F(x) < 1\}$ is the upper endpoint of $F$.

If $\omega$ is finite then Theorem 2 implies that

$$(26) \qquad\qquad \tau(X_i, X_j) \quad = \quad 0, \quad 1 \leq i < j \leq k$$

since both $X_i, X_j$ and $R_{I,p}, I = \{i, j\}$, have the same upper endpoint $\omega$. Hence the joint tail probabilities diminish faster than each of the marginal tail probability.

The next result shows that (26) holds even if $\omega = \infty$, provided that the associated random radius $R$ has a rapidly varying survival function $1 - F$, i.e.

$$(27) \qquad\qquad \lim_{t \to \infty} \frac{1 - F(ct)}{1 - F(t)} \quad = \quad 0$$

for any $c > 1$.

**Theorem 7.** *Let $\boldsymbol{X}$ be a LpGSD random vector in $\mathbb{R}^k, k \geq 2$, with the associated random radius $R$ which is almost surely positive. If the distribution function $F$ of*

*R satisfies (27), then*

$$(28) \qquad \tau(X_i, X_j/z) \;=\; 0, \quad 1 \le i < j \le k$$

*is valid for any $z \in (0, \infty)$.*

**Example 8.** [**Continue Example 4**]. Let $\boldsymbol{X} = R\boldsymbol{\mathcal{U}}$ be a $k$-dimensional Kotz Type III random vector. In view of the property (15) we have for any $c > 1$

$$\frac{\boldsymbol{P}\{R > cu\}}{\boldsymbol{P}\{R > u\}} \;=\; (1 + o(1))c^N \exp\!\big(-r[c^\delta - 1]u^\delta\big) \to 0, \quad u \to \infty.$$

This implies that the survival function $1 - F$ satisfies (27). Consequently (28) holds if $\boldsymbol{X}$ is a LpGSD random vector.

## 5. Conditional Limiting Theorems

Let $\boldsymbol{X}$ be as in Theorem 5 with $R \sim F$ such that $F$ has the upper endpoint $\omega \in (0, \infty]$. Given $I, J$ two subsets of $\{1, \ldots, k\}$ we shall derive in this section an asymptotic approximation for the distribution function of the conditional random vector $\boldsymbol{X}_I | \boldsymbol{X}_J = \boldsymbol{u}_J, \boldsymbol{u} \in \mathbb{R}^k$, letting $\boldsymbol{u}_J$ tend to some boundary point. Similar results for spherical and elliptical random vectors are derived in Hashorva (2006b,c,2007). In fact, the motivation for the aforementioned results comes from those previously reported in Berman (1992) where elliptical random vectors are discussed. As in Berman (1992) we assume a certain asymptotic tail behaviour of the distribution function $F$ related to extreme value theory. Explicitly, we shall suppose that $F$ is in the max-domain of attraction of an univariate extreme value distribution function $H$, i.e.

$$(29) \qquad \lim_{n \to \infty} \sup_{x \in \mathbb{R}} \Big| F^n(r(n)x + q(n)) - H(x) \Big| \;=\; 0,$$

where $r(n) > 0, q(n), n \ge 1$ are given constants.

We shall denote the above asymptotic relation by $F \in MDA(H)$, and refer the reader for a further insight in the extreme value theory to the following standard monographs: de Haan (1970), Leadbetter et al. (1983), Resnick (1987), Reiss (1989), Falk et al. (2004), Kotz and Nadarajah (2005).

We note in passing that $H$ is either a) the unit Gumbel distribution $\Lambda(x) = \exp(-\exp(-x))$, or b) the unit Weibull distribution $\Psi_\gamma(x) = \exp(-|x|^\gamma), x < 0, \gamma > 0$, or c) the unit Fréchet distribution $\Phi_\gamma(x) = \exp(-x^{-\gamma}), x > 0, \gamma > 0$. The symbol $\omega$ denotes again the upper endpoint of the distribution function $F$.
We consider each case separately.

**The Gumbel Case** $F \in MDA(\Lambda)$:
If $H = \Lambda$ then (29) is equivalent to the fact that there exists a positive measurable function $w$ such that

$$(30) \qquad \lim_{u \uparrow \omega} \frac{1 - F(u + x/w(u))}{1 - F(u)} \;=\; \exp(-x), \quad \forall x \in \mathbb{R}$$

is valid. The positive scaling function $w$ has the following asymptotic properties (see e.g. Resnick (1987) or Kotz and Nadarajah (2005))

$$(31) \qquad \lim_{u \uparrow \omega} \frac{w(u + x/w(u))}{w(u)} \;=\; 1$$

hold uniformly for $x$ in compact sets of $\mathbb{R}$. Furthermore

$$\lim_{u \uparrow \omega} k(u)w(u) = \infty, \tag{32}$$

with $k(u) := u$ if $\omega = \infty$ and $k(u) := \omega - u$ otherwise.

It will be shown in the next theorem that the conditional distribution of LpGSD random vectors is approximated by a Kotz Type I LpGSD random vector, provided $F$ satisfies the limiting condition (30). Evidently, the Kotz Type I LpGSD class of distributions includes the Gaussian distributions with correlation matrix equal the identity matrix. For $L_2$-norm spherical random vectors the limiting distribution is Gaussian (see Hashorva (2006b)). It is rather surprising that a large class of LpGDS distributions can be approximated by a distribution function (Kotz Type I), which is completely known, provided the associated random radius is in the Gumbel max-domain of attraction. Moreover, the limiting distribution has independent components!

**Theorem 8.** *Let $F, \boldsymbol{X}$ be as in Theorem 5 with $\omega \in (0, \infty]$ the upper endpoint of $F$, and $I, J$ be two non-empty disjoint sets of $\{1, \ldots, k\}$. Assume that distribution function $F$ is in the Gumbel max-domain of attraction with positive scaling function $w$. If $\boldsymbol{u}_n \in \mathbb{R}^k, n \geq 1$, is such that $\|\boldsymbol{u}_{n,J}\|_p < \omega, n \geq 1$, and furthermore*

$$\lim_{n \to \infty} \|\boldsymbol{u}_{n,J}\|_p = \omega, \tag{33}$$

*we then have the convergence in the distribution*

$$\left(\frac{w(\|\boldsymbol{u}_{n,J}\|_p)}{\|\boldsymbol{u}_{n,J}\|_p^{p-1}}\right)^{1/p} \left(\boldsymbol{X}_I | \boldsymbol{X}_J = \boldsymbol{u}_{n,J}\right) \xrightarrow{d} \boldsymbol{Z}, \quad n \to \infty, \tag{34}$$

*where $\boldsymbol{Z} \sim \boldsymbol{\mathcal{K}}_{\boldsymbol{\alpha}_I, p}$ is a Kotz Type I LpGSD random vector in $\mathbb{R}^{|I|}$ with parameters $\boldsymbol{\alpha}_I, N + 1 = r = s = 1$.*

For $\boldsymbol{Z} \sim \boldsymbol{\mathcal{K}}_{\boldsymbol{\alpha}_I, p}$ we have the stochastic representation

$$\boldsymbol{Z} \stackrel{d}{=} \mathcal{R}_I \boldsymbol{\mathcal{V}}_I,$$

with $\mathcal{R}_I^p > 0$ independent of $\boldsymbol{\mathcal{V}}_I$ and moreover

$$\mathcal{R}_I^p \sim Gamma(\overline{\alpha}_I, 1/p), \quad \boldsymbol{\mathcal{V}}_I \sim \mathcal{SD}(|I|, p, \boldsymbol{\alpha}_I).$$

Consequently if $p = 2$ and $\boldsymbol{\alpha} = \mathbf{1}/2 \in \mathbb{R}^k$, then $\boldsymbol{Z}$ is a standard Gaussian random vector in $\mathbb{R}^{|I|}$ with independent components. The above theorem asserts that for a spherically distributed $\boldsymbol{X}$ the conditional limiting distribution is Gaussian with the identity correlation matrix. This is shown in Corollary 3.1 of Hashorva (2006b) which is motivated by Theorem 4.1 of Berman (1983) (see also Theorem 12.4.1 in Berman (1992) and Lemma 8.2 in Berman (1982)).

It is interesting to note that the Gaussian approximation of Type I spherical random vectors ($L_2$-norm) is a special case of the Kotz approximation of LpGSD Type I random vectors. We present next an example.

**Example 9.** [**Regularly varying scaling function**] Let $\boldsymbol{X}$ be a $k$-dimensional LpGSD random vector with associated random radius $R$ which has distribution function $F$ in the Gumbel max-domain of attraction with the scaling function

$$w(u) = (1 + o(1))u^\delta L(u), \quad \delta > 0, \quad u \to \infty,$$

where $L$ is a positive function such that $\lim_{u\to\infty} L(Ku)/L(u) = 1, \forall K > 1$. Consider now positive constants $u_n, n \geq 1$ such that $\lim_{n\to\infty} u_n = \infty$ and let $I, J$ be two non-empty disjoint subsets of $\{1, \ldots, k\}$. For a given vector $\boldsymbol{a} \in \mathbb{R}^k$ such that $\boldsymbol{a}_J \neq \boldsymbol{0}_J$ and any integer $n \geq 1$ define

$$\boldsymbol{u}_n := u_n \boldsymbol{a}, \quad h_n := \left( \frac{w(u_n\|\boldsymbol{a}_J\|_p)}{(u_n\|\boldsymbol{a}_J\|_p)^{p-1}} \right)^{1/p} = (L(u_n\|\boldsymbol{a}_J\|_p))^{1/p}(u_n\|\boldsymbol{a}_J\|_p)^{(\delta+1)/p-1}.$$

Clearly, $\boldsymbol{u}_n, n \geq 1$, satisfies (33), consequently Theorem 8 implies

$$(35) \qquad\qquad h_n \boldsymbol{X}_I | \boldsymbol{X}_J = u_n \boldsymbol{a}_J \quad \overset{d}{\to} \quad \boldsymbol{Z} \sim \mathcal{K}_{\boldsymbol{\alpha}_I, p}, \quad n \to \infty,$$

where $\mathcal{K}_{\boldsymbol{\alpha}_I, p}$ denotes a standard Kotz Type I LpGSD random vector.

If $\boldsymbol{X}$ is a Kotz Type III as in Example 2, then the associated random radius has the distribution function in the Gumbel max-domain of attraction with the scaling function $w$ given by

$$w(u) = (1 + o(1))r\delta u^{\delta-1}, \quad u \to \infty.$$

This follows easily by observing that

$$\frac{\boldsymbol{P}\{R > u + x/w(u)\}}{\boldsymbol{P}\{R > u\}} \quad = \quad (1 + o(1))\left(1 + \frac{x}{r\delta u^\delta}\right)^N \exp\left(-ru^\delta\left[\left(1 + \frac{x}{r\delta u^\delta}\right)^\delta - 1\right]\right)$$

$$\to \quad \exp(-x), \quad u \to \infty.$$

Hence for this case (35) holds with $h_n := (r\delta)^{1/p}(u_n\|\boldsymbol{a}_J\|_p)^{\delta/p-1}, n \geq 1$.

**The Weibull Case $F \in MDA(\Psi_\gamma)$:**
The distribution function $F$ has necessarily a finite upper endpoint $\omega$.
Without loss of generality we assume in the following theorem that $\omega = 1$. The conditional distribution of $\boldsymbol{X}_I | \boldsymbol{X}_J = \boldsymbol{u}_{n,J}$ can be approximated ($n \to \infty$) by another LpGSD random vector as shown in the next theorem.

**Theorem 9.** *Let $F, I, J, \boldsymbol{X}, \boldsymbol{u}_n, n \geq 1$ be as in Theorem 8 and $c_n, n \geq 1$, be a sequence of positive constants converging to 0. Assume that the upper endpoint of $F$ is $\omega = 1$, and furthermore*

$$(36) \qquad\qquad \lim_{n\to\infty} \frac{1 - \|\boldsymbol{u}_{n,J}\|_p}{c_n} \quad = \quad 1$$

*holds. If $F \in MDA(\Psi_\gamma), \gamma > 0$, we then have*

$$(37) \qquad\qquad \left(\frac{1}{pc_n}\right)^{1/p}\left(\boldsymbol{X}_I | \boldsymbol{X}_J = \boldsymbol{u}_{n,J}\right) \quad \overset{d}{\to} \quad \mathcal{R}_I \boldsymbol{\mathcal{V}}_I, \quad n \to \infty,$$

*where $\mathcal{R}_I^p \sim Beta(\overline{\alpha}_I, \gamma + \overline{\alpha} - \overline{\alpha}_I - \overline{\alpha}_J), \mathcal{R}_I > 0$ and $\mathcal{R}_I$ is independent of $\boldsymbol{\mathcal{V}}_I \sim \mathcal{SD}(|I|, p, \boldsymbol{\alpha}_I)$.*

We note in passing that Theorem 12.7.1. in Berman (1992) a related result to (37) is shown for a bivariate elliptical random vector. The multivariate extension of Berman's theorem is presented in Theorem 3.2 of Hashorva (2007).

**Example 10.** [**Kummer-Beta**] Let $\boldsymbol{X}$ be as in Example 6. It follows that the random radius $R$ associated with $\boldsymbol{X}$ has the distribution function in the Weibull max-domain of attraction of $\Psi_\gamma$. Consider the sequence of vectors $\boldsymbol{u}_n = (1 -$

$1/n, 0, \ldots, 0), n \geq 1$ in $\mathbb{R}^k$. If $I = \{1, \ldots, r\}, 1 \leq r < k$, then (36) holds with $c_n = 1/n, n \geq 1$. Consequently Theorem 9 implies the convergence

$$\left(\frac{n}{p}\right)^{1/p} \left(\boldsymbol{X}_I | \boldsymbol{X}_J = \boldsymbol{u}_{n,J}\right) \quad \overset{d}{\to} \quad \mathcal{R}_I \boldsymbol{\mathcal{V}}_I, \quad n \to \infty,$$

where $J = \{r+1, \ldots, m\}, r < m \leq k, \mathcal{R}_I \sim Beta(\sum_{i=1}^r \alpha_i, \gamma + k - m)$ and $\boldsymbol{\mathcal{V}}_I \sim \mathcal{SD}(r, p, \boldsymbol{\alpha}_I)$.

**The Fréchet Case** $F \in MDA(\Phi_\gamma)$:
In this case $F$ has an infinite upper endpoint. Similarly to the two other cases of max-domain of attraction it is possible also to approximate here the conditional distribution of LpGSD random vectors. We have:

**Theorem 10.** *Let $F, I, J, \boldsymbol{X}, c_n, \boldsymbol{u}_n, n \geq 1$ be as in Theorem 9 such that*

$$(38) \qquad \lim_{n \to \infty} c_n \boldsymbol{u}_{n,J} = \boldsymbol{u}_J \neq \boldsymbol{0}_J$$

*is valid. If $F \in MDA(\Phi_\gamma), \gamma > 0$, we then have the convergence in the distribution*

$$(39) \qquad c_n \left(\boldsymbol{X}_I | \boldsymbol{X}_J = \boldsymbol{u}_{n,J}\right) \overset{d}{\to} \boldsymbol{Y}_I | \boldsymbol{Y}_J = \boldsymbol{u}_J, \quad n \to \infty,$$

*where $\boldsymbol{Y} \sim \mathcal{GSD}(k, p, \boldsymbol{\alpha}, F_\gamma)$ with the distribution function $F_\gamma$ defined by*

$$F_\gamma(r) := 1 - (r/\|\boldsymbol{u}_J\|_p)^{-\gamma - p(\overline{\alpha} - \overline{\alpha}_I - \overline{\alpha}_J)}, \quad \forall r \geq \|\boldsymbol{u}_J\|_p.$$

A natural choice for the constants $c_n, n \geq 1$ in the above theorem is $c_n := 1/\|\boldsymbol{u}_{n,J}\|_p, n \geq 1$, provided that $\lim_{n \to \infty} \|\boldsymbol{u}_{n,J}\|_p = \infty$. The latter is actually a necessary condition for (38) to hold.

**Example 11.** [**Kotz Type III**] Let $\boldsymbol{X}$ be as in Example 4 with $N < 0, \delta \leq 0, p > 0$. Then the associated random radius $R$ of $\boldsymbol{X}$ has the distribution function in the max-domain of attraction of the Fréchet distribution $\Phi_{-N}$. Consequently, if $\boldsymbol{X}$ is also a LpGSD random vector, Theorem 10 implies for $I, J$ disjoint index sets and $u_n > 0, n \geq 1$ such that $\lim_{n \to \infty} u_n = \infty$

$$\frac{1}{u_n} \left(\boldsymbol{X}_I | \boldsymbol{X}_J = u_n \boldsymbol{u}_J\right) \overset{d}{\to} \boldsymbol{Y}_I | \boldsymbol{Y}_J = \boldsymbol{u}_J, \quad n \to \infty,$$

where $\|\boldsymbol{u}_J\|_p > 0$ and $\boldsymbol{Y} \sim \mathcal{GSD}(k, p, \boldsymbol{\alpha}, F_{-N})$, with the distribution function $F_{-N}$ given by

$$F_{-N}(r) = 1 - (r/\|\boldsymbol{u}_J\|_p)^{N - p(k-m)}, \quad \forall r \geq \|\boldsymbol{u}_J\|_p.$$

## 6. Tail Asymptotics

Let $\boldsymbol{X}$ be a $k$-dimensional LpGDS random vector with the associated random radius $R$. The distributional properties of $\boldsymbol{X}$ are determined by those of $R$. Similarly we expect that in an asymptotic context the asymptotic behaviour of $\boldsymbol{P}\{\boldsymbol{X}/n \in B\}, n \to \infty$, with $B$ being a Borel set, is defined by the tail asymptotics of $R$. In the special cases when $R$ has distribution function in the max-domain of attraction of an univariate extreme value distribution $H$, then the tail asymptotic behaviour of $X_1$ can be determined by applying Lemma 16 in the Appendix. The case where $R$ is regularly varying with index $\gamma \geq 0$, i.e.

$$\lim_{t \to \infty} \frac{\boldsymbol{P}\{R > tx\}}{\boldsymbol{P}\{R > t\}} \quad = \quad x^{-\gamma}, \quad \forall x > 0,$$

is quite tractable as shown in Hashorva (2006a). The above asymptotic relation defines the tail asymptotic of $\boldsymbol{X}$ and in particular of its components, and moreover the converse is true.

Indeed we have the following result:

**Theorem 11.** *Let $\boldsymbol{X} \sim \mathcal{GSD}(k, p, \boldsymbol{\alpha}, F)$ be a $k$-dimensional random vector with the associated random radius $R$, and $A \in \mathbb{R}^{k \times k}$ be a non-singular matrix. The statements below are then equivalent:*

*i) $|X_1|$ is regularly varying with a positive index $\gamma$.*

*ii) For any non-empty $I \subset \{1, \ldots, k\}$ the random radius $\|\boldsymbol{X}_I\|_p$ is regularly varying with index $\gamma > 0$, and furthermore if $|I| < k$ then*

$$(40)\, \boldsymbol{P}\{\|\boldsymbol{X}_I\|_p > u\} \quad = \quad (1 + o(1)) \frac{\Gamma(\overline{\alpha})\Gamma(\overline{\alpha}_I + \gamma/p)}{\Gamma(\overline{\alpha}_I)\Gamma(\overline{\alpha} + \gamma/p)} \boldsymbol{P}\{\|\boldsymbol{X}\|_p > u\}, \quad u \to \infty.$$

*iii) For any non-empty $I \subset \{1, \ldots, k\}$ with $m$ elements and any Borel set $B \subset \mathbb{R}^m$ not containing the origin $\boldsymbol{0} \in \mathbb{R}^m$*

$$\lim_{u \to \infty} \frac{\boldsymbol{P}\{(A_{II}\boldsymbol{X}_I + \boldsymbol{\mu}_I)/u \in B\}}{\boldsymbol{P}\{X_i > u\}}$$

$$(41) \qquad = \quad \frac{2\gamma\Gamma(\alpha_i)\Gamma(\overline{\alpha} + \gamma/p)}{\Gamma(\overline{\alpha})\Gamma(\alpha_i + \gamma/p)} \int_0^\infty \boldsymbol{P}\{rA_{II}\boldsymbol{\mathcal{V}}_I \in B\} r^{-\gamma-1}\, dr$$

*holds with $i \le k$, $\boldsymbol{\mu} \in \mathbb{R}^k$ and $\boldsymbol{\mathcal{V}}_I \sim \mathcal{SD}(m, p, \boldsymbol{\alpha}_I)$.*

**Corollary 12.** *Let $A, R, \boldsymbol{X}$ be as in Theorem 11. Assume that the associated random radius $R$ or the first components $X_1$ of $\boldsymbol{X}$ is regularly varying with positive index $\gamma$. Let $X_i \sim G_i, i \le k, a_i(n) := G_i^{-1}(1 - 1/n), n > 1$ with $G_i^{-1}$ the generalised inverse of $G_i$ and set $\boldsymbol{Y} := A\boldsymbol{X}$. Then we have for any $\boldsymbol{y} = (y_1, \ldots, y_k)^\top > \boldsymbol{0}$*

$$\lim_{n \to \infty} \boldsymbol{P}\{Y_1/a_1(n) \le y_1, \ldots, Y_k/a_k(n) \le y_k\}^n$$

$$(42) \qquad = \quad \exp\Big(-\gamma \int_0^\infty \boldsymbol{P}\{r\boldsymbol{c}A\boldsymbol{\mathcal{U}} \not\le \boldsymbol{y}\} r^{-\gamma-1}\, dr\Big),$$

*where the vector $\boldsymbol{c} = (c_1, \ldots, c_k)^\top$ has components given by*

$$c_i := \Big( \frac{\Gamma(\overline{\alpha})\Gamma(\alpha_i + \gamma/p)}{2\Gamma(\alpha_i)\Gamma(\overline{\alpha} + \gamma/p)} \Big)^{-1/\gamma}, \quad i \le k.$$

**Remark 2.** i) Statement *iii*) in Theorem 11 implies that $\boldsymbol{X}$ is a multivariate regularly varying random vector in $\mathbb{R}^k$ with index $\gamma > 0$, and in particular $|X_i|, i \le k$ is regularly varying with index $\gamma$. See Basrak at al. (2002) for more details on regular variation of random vectors.

ii) If $A$ is the identity matrix then the right-hand side of (42) is a distribution function with the Fréchet marginal distributions $\Phi_\gamma(x) = \exp(-x^{-\gamma}), x > 0$.

iii) Using (41) we obtain for any $\delta, \lambda, p, \gamma$ positive with $\lambda - \delta > 0$

$$\frac{\gamma\Gamma(\delta)\Gamma(\lambda + \gamma/p)}{\Gamma(\lambda)\Gamma(\delta + \gamma/p)} \int_1^\infty \boldsymbol{P}\{Z > r^{-p}\} r^{-\gamma-1}\, dr = 1,$$

with $Z \sim Beta(\delta, \lambda - \delta)$. Consequently we have

$$h_{\delta,\lambda,\gamma}(x) \quad := \quad \frac{\gamma\Gamma(\delta)\Gamma(\lambda + \gamma/p)}{\Gamma(\lambda)\Gamma(\delta + \gamma/p)} \int_x^\infty \boldsymbol{P}\{Z > r^{-p}\} r^{-\gamma-1}\, dr, \quad x \ge 1$$

is a survival function of a positive random variable in $[1, \infty)$.

**Example 12.** [**Kotz Type II**] Let as in Example 3 $\boldsymbol{X}$ be a Kotz Type II LpGSD random vector with parameters $\boldsymbol{\alpha}, s < 0, N < -\overline{\alpha}$. It follows that $R$ is regularly varying with the index $-p(N+\overline{\alpha})$. Hence, any component of $\boldsymbol{X}$ is regularly varying with the index $-p(N + \overline{\alpha})$.

## 7. Proofs

Proof of Theorem 1: We carry out the following transformations of variables

$$y_i = x_i r^{-p}, \quad i = 1, \ldots, k-1, \text{ and } r^p = \sum_{i=1}^{k} |x_i|^p.$$

Calculating the Jacobian of this transformation we arrive at the p.d.f of $\boldsymbol{X}$

$$
\begin{aligned}
h(r, z_1, \ldots, z_{k_1}) &= 2g(r^p) r^{p \sum_{i=1}^{k} \alpha_i - 1} \prod_{i=1}^{k-1} |z_i|^{p\alpha_i - 1} \Big(1 - \sum_{i=1}^{k-1} |z_i|^p \Big)^{\alpha_k - 1} \\
&= \frac{2^k \prod_{i=1}^{k} \Gamma(\alpha_i)}{p^{k-1} \Gamma(\overline{\alpha})} g(r^p) r^{p \sum_{i=1}^{k} \alpha_i - 1} c \prod_{i=1}^{k-1} |z_i|^{p\alpha_i - 1} \Big(1 - \sum_{i=1}^{k-1} |z_i|^p \Big)^{\alpha_k - 1},
\end{aligned}
$$

with $c^{-1} := (2/p)^{k-1} \prod_{i=1}^{k} \Gamma(\alpha_i)/\Gamma(\overline{\alpha})$. Hence the result follows by recalling the form of density function in (8).

Now, if $\boldsymbol{X}$ has p.d.f $h$ given by (6), then in view of (1) the p.d.f of $R\boldsymbol{\mathcal{U}}$ is given by

$$h(r, u_1, \ldots, u_{k-1}) = f(r) q(u_1, \ldots, u_{k-1}),$$

with $q$ as in (2). Transforming the variables as above it follows that $\boldsymbol{X}$ has p.d.f $h$ given by (6), hence the proof is complete.

$\square$

Proof of Theorem 2: The proof is analogous (considering $p$ instead of 2) to the proof of Theorem 4.1 of Fang and Fang (1990). For the sake of completeness we shall provide a sketch. First note that $\boldsymbol{X} \overset{d}{=} R\boldsymbol{\mathcal{U}}$ with $R$ independent of $\boldsymbol{\mathcal{U}} \sim \mathcal{SD}(k, p, \boldsymbol{\alpha})$. The properties of $\boldsymbol{\mathcal{U}}$ and in particular can be derived considering $\boldsymbol{X} \sim \mathcal{K}_{\boldsymbol{\alpha},p}$ as in Example 2 ($N + 1 = r = s = 1$). Since also $\boldsymbol{X}_{I_j}, 1 \leq j \leq m$ are LpGSD random vectors we have

$$\boldsymbol{X}_{I_j} \overset{d}{=} R_{I_j} \boldsymbol{\mathcal{V}}_{I_j}, \quad 1 \leq j \leq m,$$

with $R_{I_j} \overset{d}{=} \|\boldsymbol{X}_{I_j}\|_p$ independent of $\boldsymbol{\mathcal{V}}_{I_j} \sim \mathcal{SD}(|I_j|, p, \boldsymbol{\alpha}_{I_j})$.

By Lemma 13 and (13) we have

$$R^p \overset{d}{=} \|\boldsymbol{X}\|_p^p = \Gamma(\overline{\alpha}, 1/p), \quad R_{I_j}^p \overset{d}{=} \|\boldsymbol{X}_{I_j}\|_p^p = \Gamma(a_j, 1/p), \quad 1 \leq j \leq m,$$

with $a_j := \sum_{i \in I_j} \alpha_i$. Denote $\boldsymbol{a}_m := (a_1, \ldots, a_m)^\top \in (0, \infty)^m$ and

$$\boldsymbol{Z}_j := \frac{\boldsymbol{X}_{I_j}}{\|\boldsymbol{X}_{I_j}\|_p}, \quad \boldsymbol{W}_m := (W_1, \ldots, W_m)^\top, \text{ with } W_j := \frac{\|\boldsymbol{X}_{I_j}\|_p}{\|\boldsymbol{X}\|_p}, 1 \leq j \leq m.$$

Evidently, $\|\boldsymbol{W}\|_p = 1$ almost surely. The proof now follows easily since $\boldsymbol{W}_m \sim \mathcal{SD}(m, p, \boldsymbol{a}_m)$.

$\square$

PROOF OF COROLLARY 3: Let $J = \{1, \dots, k\} \setminus I$, and $F$ denote the distribution function of $RW$ where $W > 0$ is independent of $R$ with $W \sim Beta(\overline{\alpha}_I, \overline{\alpha}_J)$. Applying Theorem 2 to partitions $I, J$ we obtain

$$\boldsymbol{X}_I \stackrel{d}{=} R_{I,p} \boldsymbol{\mathcal{V}}_I, \text{ with } \boldsymbol{\mathcal{V}}_I \sim \mathcal{SD}(|I|, p, \boldsymbol{\alpha}_I).$$

Since $R_{I,p}$ independent of $\boldsymbol{\mathcal{U}}$, $\boldsymbol{X}_I$ is a LpGSD random vector in $\mathbb{R}^{|I|}$. Using Lemma 13 we have

$$R_{I,p} = \|\boldsymbol{X}_I\|_p \stackrel{d}{=} \|RW\boldsymbol{\mathcal{V}}_I\|_p = RW\|\boldsymbol{\mathcal{V}}_I\|_p = RW,$$

and the proof is completed.

$\square$

PROOF OF COROLLARY 4: Let $\mathcal{I}_i, i \leq k$ be independent random variables taking values $-1, 1$ with probability $1/2$. For simplicity we show the proof when $k = 3$. The general case $k > 3$ follows utilising similar arguments. By the assumption $\boldsymbol{X} \stackrel{d}{=} R\boldsymbol{\mathcal{U}}$, with $\boldsymbol{\mathcal{U}} \sim \mathcal{SD}(3, p, (\alpha_1, \alpha_2, \alpha_3))$ independent of $R$. In view of Theorem 2, we have

$$\boldsymbol{X} \stackrel{d}{=} R\Big(\mathcal{I}_1(1 - V_1^p)^{1/p}, \mathcal{I}_2 V_1 (1 - V_2^p)^{1/p}, \mathcal{I}_3 V_1 V_2\Big),$$

where $V_1^p \sim Beta(\alpha_2 + \alpha_3, \alpha_1), V_2^p \sim Beta(\alpha_3, \alpha_2)$. Define the random angles $\Theta_1 \in [0, \pi], \Theta_2 \in [0, 2\pi]$ such that

$$\sin(\Theta_1) := V_1^{p/2}, \quad |\cos(\Theta_1)|^{2/p} := (1 - V_1^p)^{1/p},$$

$$|\cos(\Theta_2)|^{2/p} := (1 - V_2^p)^{1/p}, \quad |\sin(\Theta_2)| := V_2^{p/2},$$

and $\text{sign}(\cos(\Theta_2))$ independent of $\text{sign}(\sin(\Theta_2))$ two symmetric random variables. It follows that the density function of $\Theta_1$ is given by

$$q_1(\theta) := \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}{\Gamma(\alpha_2 + \alpha_3)\Gamma(\alpha_1)} |\sin(\theta)|^{2(\alpha_2 + \alpha_3) - 1} |\cos(\theta)|^{2\alpha_1 - 1}, \quad \theta \in [0, \pi],$$

and $\Theta_2$ has density function

$$q_2(\theta) := \frac{1}{2} \frac{\Gamma(\alpha_2 + \alpha_3)}{\Gamma(\alpha_3)\Gamma(\alpha_2)} |\sin(\theta)|^{2\alpha_3 - 1} |\cos(\theta)|^{2\alpha_2 - 1}, \quad \theta \in [0, 2\pi].$$

Hence we have the stochastic representation

$$X_1 \stackrel{d}{=} R|\cos(\Theta_1)|^{2/p} \text{sign}(\cos(\Theta_1)), \quad X_2 \stackrel{d}{=} R[\sin(\Theta_1)|\cos(\Theta_2)|]^{2/p} \text{sign}(\cos(\Theta_2)),$$

$$X_3 \stackrel{d}{=} R[\sin(\Theta_1)|\sin(\Theta_2)|]^{2/p} \text{sign}(\sin(\Theta_2)).$$

The converse follows easily by reversing the argument.

$\square$

PROOF OF THEOREM 5: The proof is based on the stochastic representation (1) and the amalgamation property with respect to the partitions $I, J$. It follows along the lines of the proof of Theorem 5 in Cambanis et al. (1981).

$\square$

PROOF OF THEOREM 6: The amalgamation property of LpGSD random vectors shows that the conditional and marginal distributions of $L_p$-norm generalised symmetrised Dirichlet random vectors are of the same form as those for the case of $L_2$-norm. Thus the proof of the general case $p > 0$ follows by utilising the same arguments as in the proof of Theorem 4.3 in Fang and Fang (1990).

$\Box$

PROOF OF THEOREM 7: Let $i, j, i \neq j$ be given and $z > 0, c_0 \in (0, 1)$ be constants. Set $k_z^p := \inf\{|x_1|^p + |x_2|^p : x_1 \geq 1, x_2 \geq z\} \geq 1$ which does exist. In view of Corollary 3, we obtain for any $t \in (0, \omega)$ (write $I := \{i, j\}$)

$$\frac{\boldsymbol{P}\{X_i > t, X_j > tz\}}{\boldsymbol{P}\{X_i > t\} + \boldsymbol{P}\{X_j > tz\}} \quad \leq \quad \frac{\boldsymbol{P}\{|X_i|^p + |X_j|^p \geq k_z t^p\}}{\boldsymbol{P}\{X_i > t\}}$$

$$= \quad \frac{2\boldsymbol{P}\{R_{I,p} \geq k_z t\}}{\boldsymbol{P}\{|X_i| > t\}}$$

$$\leq \quad \frac{2\boldsymbol{P}\{R \geq k_z t\}}{\boldsymbol{P}\{RW_i > t, W_1 > c_0\}}$$

$$\leq \quad \frac{2\boldsymbol{P}\{R \geq k_z t\}}{\boldsymbol{P}\{R > t/c_0\}\boldsymbol{P}\{W_i > c_0\}},$$

with $W_i > 0$ almost surely such that $W_i^p \sim Beta(\alpha_i, \overline{\alpha} - \alpha_i)$. Since the survival function of the random radius $R$ is rapidly varying and $k_z \geq 1$, the claim follows by choosing $c_0 \in (1/k_z, 1)$ and then letting $t \to \infty$.

$\Box$

PROOF OF THEOREM 8: Let $\omega \in (0, \infty]$ denote the upper endpoint of the distribution function $F$ and set

$$a_n := \|\boldsymbol{u}_{n,J}\|_p, \quad \text{and } w_n := w(a_n), \qquad n \geq 1.$$

By the assumptions $\lim_{n\to\infty} a_n = \omega$ and $F \in MDA(\Lambda)$ with a positive scaling function $w$, we thus obtain that in view of (31) and (32)

$$(43) \qquad \lim_{n\to\infty} a_n w_n = \infty, \quad \lim_{n\to\infty} w_n(\omega - a_n) = \infty.$$

Theorem 5 and Lemma 16 imply that $\boldsymbol{X}_{I \cup J}$ is a $LpGSD$ random vector with the associated random radius $R^*$ which has distribution function in the max-domain of attraction of $\Lambda$ and the scaling function $w$. Therefore, we may assume for simplicity $I \cup J = \{1, \ldots, k\}$. In view of Theorem 5, we have for any large $n$

$$(44) \qquad \left(\boldsymbol{X}_I | \boldsymbol{X}_J = \boldsymbol{u}_{n,J}\right) \stackrel{d}{=} R_{a_n} \boldsymbol{V}_I, \quad \boldsymbol{V}_I \sim \mathcal{SD}(|I|, p, \boldsymbol{\alpha}_I),$$

with $\boldsymbol{V}_I$ independent of $R_{a_n}$ such that

$$\boldsymbol{P}\{R_{a_n} > x\} = \frac{\int_{(a_n^p + x^p)^{1/p}}^{\omega} (r^p - a_n^p)^{\overline{\alpha}_I - 1} r^{-p\overline{\alpha} + p} \, dF(r)}{\int_{a_n}^{\omega} (r^p - a_n^p)^{\overline{\alpha}_I - 1} r^{-p\overline{\alpha} + p} \, dF(r)}$$

for all $x \in (0, (\omega^p - a_n^p)^{1/p})$. Furthermore, (30) implies that for any $s \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{1 - F(a_n + s/w_n)}{1 - F(a_n)} = \exp(-s).$$

Hence the sequence of distribution functions

$$F_n(s) := \frac{F(a_n + s/w_n) - F(a_n)}{1 - F(a_n)}, \quad s \geq 0, n \geq 1$$

converges uniformly to the unit exponential distribution as $n \to \infty$. Moreover (43) implies for any $x \geq 0$

$$(x^p a_n^{p-1}/w_n + a_n^p)^{1/p} = a_n + (1 + o(1))p^{-1} x^p/w_n, \quad n \to \infty.$$

Transforming the variables we have for $n$ large

$$
\begin{aligned}
& \boldsymbol{P}\{R_{a_n} > x(a_n^{p-1}/w_n)^{1/p}\} \\
& = \frac{\int_{a_n+(1+o(1))p^{-1}x^p/w_n}^{\omega}(r^p - a_n^p)^{\overline{\alpha}_I-1}r^{-p\overline{\alpha}+p}\,dF(r)}{\int_{a_n}^{\omega}(r^p - a_n^p)^{\overline{\alpha}_I-1}r^{-p\overline{\alpha}+p}\,dF(r)} \\
& = \frac{\int_{x^p/p}^{w_n(\omega-a_n)}((a_n + s/w_n)^p - a_n^p)^{\overline{\alpha}_I-1}(a_n + s/w_n)^{-p\overline{\alpha}+p}\,dF_n(s)}{\int_0^{w_n(\omega-a_n)}((a_n + s/w_n)^p - a_n^p)^{\overline{\alpha}_I-1}(a_n + s/w_n)\,dF_n(s)} \\
& = \frac{\int_{x^p/p}^{w_n(\omega-a_n)} s^{\overline{\alpha}_I-1}(1 + o(1))\,dF_n(s)}{\int_0^{w_n(\omega-a_n)} s^{\overline{\alpha}_I-1}(1 + o(1))\,dF_n(s)}, \quad n \to \infty.
\end{aligned}
$$

Lemma 4.4 of Hashorva (2006b) implies that

$$
\lim_{n\to\infty} \int_{x^p/p}^{w_n(\omega-a_n)} s^{\overline{\alpha}_I-1}\,dF_n(s) = \int_{x^p/p}^{\infty} s^{\overline{\alpha}_I-1}\exp(-s)\,ds, \quad \forall x \geq 0.
$$

Hence we obtain for any $x > 0$

$$
\lim_{n\to\infty} \boldsymbol{P}\{R_{a_n} > x(a_n^{p-1}/w_n)^{1/p}\} = \frac{1}{\Gamma(\overline{\alpha}_I)} \int_{x^p/p}^{\infty} s^{\overline{\alpha}_I-1}\exp(-s)\,ds =: \boldsymbol{P}\{\mathcal{R}_I > x\}.
$$

Consequently we have the convergence in distribution

$$
\left(\frac{w_n}{a_n^{p-1}}\right)^{1/p} R_{a_n} \quad \xrightarrow{d} \quad \mathcal{R}_I, \quad n \to \infty,
$$

where $\mathcal{R}_I > 0$ such that $\mathcal{R}_I^p \sim \Gamma(\overline{\alpha}_I, 1/p)$. Noting that $\mathcal{R}_I$ is independent of $\boldsymbol{\mathcal{U}}$ we arrive at the desired result. $\qquad\square$

PROOF OF THEOREM 9: Let $a_n, R_{a_n}, n \geq 1$ be as in the proof of Theorem 8 and set $F_n(s) := F(1-c_n s), s \geq 0, n \geq 1$. For simplicity we assume that $a_n = 1 - c_n, n \geq 1$ and denote $h_n := (1 - F(a_n))^{-1}, n \geq 1$. If $I \cup J$ has less then $k$ elements then Theorem 5 and Lemma 16 (Appendix) imply that the random vector $\boldsymbol{X}_{I\cup J}$ is a LpGSD random vector with associated random radius in the max-domain of attraction of Weibull distribution $\Psi_{\gamma^*}, \gamma^* := \gamma + \overline{\alpha} - \overline{\alpha}_I - \overline{\alpha}_J > 0$. For simplicity, we consider therefore the case that $I \cup J = \{1, \ldots, k\}$ only. Since $F \in MDA(\Psi_\gamma)$ we have (see Kotz and Nadarajah (2005))

$$
\lim_{n\to\infty} h_n[1 - F_n(s)] = s^\gamma, \quad \forall s > 0.
$$

Furthermore $\lim_{n\to\infty} c_n = 0$ implies that

$$
(pc_n x^p + (1 - c_n)^p)^{1/p} = 1 - c_n(1 - x^p)(1 + o(1)), \quad n \to \infty.
$$

Transforming the variables we obtain

$$
\begin{aligned}
\boldsymbol{P}&\{R_{a_n} > x(pc_n)^{1/p}\}\\
&= \frac{\int_{(pc_n x^p + (1-c_n)^p)^{1/p}}^{1}(r^p - (1-c_n)^p)^{\overline{\alpha}_I - 1} r^{-p\overline{\alpha}+p}\, dF(r)}{\int_{1-c_n}^{1}(r^p - (1-c_n)^p)^{\overline{\alpha}_I - 1} r^{-p\overline{\alpha}+p}\, dF(r)}\\
&= \frac{\int_{1-c_n(1-x^p)(1+o(1))}^{1}(r^p - (1-c_n)^p)^{\overline{\alpha}_I - 1} r^{-p\overline{\alpha}+p}\, dF(r)}{\int_{1-c_n}^{1}(r^p - (1-c_n)^p)^{\overline{\alpha}_I - 1} r^{-p\overline{\alpha}+p}\, dF(r)}\\
&= \frac{\int_{0}^{(1-x^p)(1+o(1))}(1 - s)^{\overline{\alpha}_I - 1}(1+o(1))\, d(h_n F_n(s))}{\int_{0}^{1}(1 - s)^{\overline{\alpha}_I - 1}(1+o(1))\, d(h_n F_n(s))}.
\end{aligned}
$$

Utilising similar arguments as in Theorem 3.2 in Hashorva (2007) we have

$$
\begin{aligned}
\lim_{n\to\infty} \boldsymbol{P}\{R_{a_n} > x(pc_n)^{1/p}\} &= \frac{\int_{0}^{1-x^p}(1-s)^{\overline{\alpha}_I - 1} s^{\gamma - 1}\, ds}{\int_{0}^{1}(1-s)^{\overline{\alpha}_I - 1} s^{\gamma - 1}\, ds}\\
&= 1 - \frac{\int_{0}^{x^p} s^{\overline{\alpha}_I - 1}(1-s)^{\gamma - 1}\, ds}{\int_{0}^{1} s^{\overline{\alpha}_I - 1}(1-s)^{\gamma - 1}\, ds}.
\end{aligned}
$$

We thus have the convergence in the distribution

$$
\left(\frac{1}{pc_n}\right)^{1/p} R_{a_n} \quad \overset{d}{\to} \quad \mathcal{R}_I, \quad n \to \infty,
$$

where $\mathcal{R}_I$ satisfies $\mathcal{R}_I^p \sim Beta(\overline{\alpha}_I, \gamma)$ almost surely. Now, the proof follows using (44) and the fact that $\mathcal{R}_I$ is independent of $\boldsymbol{\mathcal{U}}$.

$\square$

PROOF OF THEOREM 10: Let $a_n, R_{a_n}, n \geq 1$, be as in the proof of Theorem 8 and set

$$
F_\gamma(r) := 1 - \left(\frac{r}{\|\boldsymbol{u}_J\|_p}\right)^{-\gamma - p(\overline{\alpha} - \overline{\alpha}_I - \overline{\alpha}_J)}, \quad \forall r \geq \|\boldsymbol{u}_J\|_p.
$$

Evidently, $F_\gamma$ is a distribution function on $[\|\boldsymbol{u}_J\|_p, \infty)$. In view of Theorem 5 and Lemma 16 we need to show the claim only for the case $I \cup J = \{1, \ldots, k\}$. Assume now for simplicity that $\boldsymbol{u}_{n,J} = d_n \boldsymbol{u}_J, d_n := 1/c_n, n \geq 1$. The upper endpoint of the distribution function $F$ is $\infty$ by the assumption. For any $x > 0$ and large $n$ we have

$$
\begin{aligned}
\boldsymbol{P}\{R_{a_n} > d_n x\} &= \frac{\int_{d_n(\|\boldsymbol{u}_J\|_p^p + x^p)^{1/p}}^{\infty}(r^p - d_n^p \|\boldsymbol{u}_J\|_p^p)^{\overline{\alpha}_I - 1} r^{-p\overline{\alpha}+p}\, dF(r)}{\int_{d_n \|\boldsymbol{u}_J\|_p}^{\infty}(r^p - d_n^p \|\boldsymbol{u}_J\|_p^p)^{\overline{\alpha}_I - 1} r^{-p\overline{\alpha}+p}\, dF(r)}\\
&= \frac{\int_{(\|\boldsymbol{u}_J\|_p^p + x^p)^{1/p}}^{\infty}(r^p - \|\boldsymbol{u}_J\|_p^p)^{\overline{\alpha}_I - 1} r^{-p\overline{\alpha}+p}\, d(h_n F(d_n r))}{\int_{\|\boldsymbol{u}_J\|_p}^{\infty}(r^p - \|\boldsymbol{u}_J\|_p^p)^{\overline{\alpha}_I - 1} r^{-p\overline{\alpha}+p}\, d(h_n F(d_n r))},
\end{aligned}
$$

with $h_n := (1 - F(d_n))^{-1}, n \geq 1$. Since $\lim_{n\to\infty} d_n = \infty$, the assumption on $F$ implies

$$
\lim_{n\to\infty} h_n[1 - F(d_n x)] = x^{-\gamma}, \quad \forall x > 0,
$$

hence we have by Fatou Lemma (see e.g. Kallenberg (1997))

$$\liminf_{n\to\infty} \int_{(\|\boldsymbol{u}_J\|_p^p+x^p)^{1/p}}^{\infty} (r^p - \|\boldsymbol{u}_J\|_p^p)^{\overline{\alpha}_I-1} r^{-p\overline{\alpha}+p} \, d(h_n F(d_n r))$$

$$\geq \int_{(\|\boldsymbol{u}_J\|_p^p+x^p)^{1/p}}^{\infty} (r^p - \|\boldsymbol{u}_J\|_p^p)^{\overline{\alpha}_I-1} r^{-p\overline{\alpha}+p} \, d(r^{-\gamma}).$$

It follows by the Karamata Theorem (see e.g. Resnick (1987))

$$\limsup_{n\to\infty} \int_{(\|\boldsymbol{u}_J\|_p^p+x^p)^{1/p}}^{\infty} (r^p - \|\boldsymbol{u}_J\|_p^p)^{\overline{\alpha}_I-1} r^{-p\overline{\alpha}+p} \, d(h_n F(d_n r))$$

$$\leq \int_{(\|\boldsymbol{u}_J\|_p^p+x^p)^{1/p}}^{\infty} (r^p - \|\boldsymbol{u}_J\|_p^p)^{\overline{\alpha}_I-1} r^{-p\overline{\alpha}+p} \, d(r^{-\gamma}).$$

Consequently

$$\lim_{n\to\infty} \boldsymbol{P}\{R_{a_n} > d_n x\}$$

$$= \frac{\int_{(\|\boldsymbol{u}_J\|_p^p+x^p)^{1/p}}^{\infty} (r^p - \|\boldsymbol{u}_J\|_p^p)^{\overline{\alpha}_I-1} r^{-p(\overline{\alpha}_I+\overline{\alpha}_J)+p} r^{-p(\overline{\alpha}-\overline{\alpha}_I-\overline{\alpha}_J)} \, d(r^{-\gamma})}{\int_{\|\boldsymbol{u}_J\|_p}^{\infty} (r^p - \|\boldsymbol{u}_J\|_p^p)^{\overline{\alpha}_I-1} r^{-p(\overline{\alpha}_I+\overline{\alpha}_J)+p} r^{-p(\overline{\alpha}-\overline{\alpha}_I-\overline{\alpha}_J)} \, d(r^{-\gamma})}$$

$$= \frac{\int_{(\|\boldsymbol{u}_J\|_p^p+x^p)^{1/p}}^{\infty} (r^p - \|\boldsymbol{u}_J\|_p^p)^{\overline{\alpha}_I-1} r^{-p(\overline{\alpha}_I+\overline{\alpha}_J)+p} \, dF_\gamma(r)}{\int_{\|\boldsymbol{u}_J\|_p}^{\infty} (r^p - \|\boldsymbol{u}_J\|_p^p)^{\overline{\alpha}_I-1} r^{-p(\overline{\alpha}_I+\overline{\alpha}_J)+p} \, dF_\gamma(r)}$$

$$=: \boldsymbol{P}\{\mathcal{R}_I > x\}.$$

We note that

$$(r^p - \|\boldsymbol{u}_J\|_p^p)^{\overline{\alpha}_I-1} r^{-p\overline{\alpha}+p} \leq r^{p\overline{\alpha}_I-p} r^{-p\overline{\alpha}+p} \leq r^{-p\overline{\alpha}_J} \leq 1, \quad \forall r > 0.$$

Applying now (44) and recalling that the random variable $R_{a_n}, n \geq 1$ is independent of $\boldsymbol{\mathcal{U}}$, we arrive at

$$c_n\Big(\boldsymbol{X}_I | \boldsymbol{X}_J = \boldsymbol{u}_{n,J}\Big) \quad \overset{d}{\to} \quad \mathcal{R}_I \boldsymbol{\mathcal{V}}_I, \quad n \to \infty.$$

In view of Theorem 5 we have

$$\mathcal{R}_I \boldsymbol{\mathcal{V}}_I \quad \overset{d}{=} \quad \boldsymbol{Y}_I | \boldsymbol{Y}_J = \boldsymbol{u}_J,$$

where $\boldsymbol{Y} \sim \mathcal{GSD}(k, p, \boldsymbol{\alpha}, F_\gamma)$. This completes the proof.

$\square$

PROOF OF THEOREM 11: We shall show that statement i) implies that $R$ is regularly varying with index $\gamma > 0$. The rest of the proof follows along the lines of Theorem 3.1 of Hashorva (2006a).
Let $\boldsymbol{Z} := (Z_1, \ldots, Z_k)^\top$ be a Kotz Type I LpGSD random vector in $\mathbb{R}^k$ with coefficient $\boldsymbol{\alpha}$ and $V > 0$ be a random variable such that $V \sim Beta(\alpha_1, \overline{\alpha} - \alpha_1)$. Assume that $V, \boldsymbol{Z}$ and $\boldsymbol{X}$ are mutually independent and denote

$$R := \|\boldsymbol{X}\|_p = \Big(\sum_{j=1}^k |X_i|^p\Big)^{1/p}, \quad \widetilde{R} := \|\boldsymbol{Z}\|_p = \Big(\sum_{j=1}^k |Z_i|^p\Big)^{1/p}.$$

By (13) $\widetilde{R}^p \sim Gamma(\overline{\alpha}, 1/p)$ Since $X_1$ is symmetric about 0, the assumptions that $X_1$ is regularly varying with index $\gamma$ implies that $|X_1|^p$ is also regularly varying with index $\gamma/p > 0$. $\widetilde{R}$ is independent of $|X_1|$, hence applying Lemma 17 (see

Appendix) we have that $(\widetilde{R} |X_1|)^p$ is also regular varying with positive index $\gamma/p$. In view of Corollary 3 we have

$$(\widetilde{R} |X_1|)^p \stackrel{d}{=} \widetilde{R}^p (R^p V) \stackrel{d}{=} (\widetilde{R}^p V) R^p \stackrel{d}{=} (|Z_1|R)^p,$$

consequently $(|Z_1|R)^p$ is regularly varying with index $\gamma/p$.

Applying once more (13) we have $|Z_1|^p \sim Gamma(\alpha_1, 1/p)$ with $|Z_1|^p$ being independent of $R^p$. Lemma 17 implies that $R^p$ is regularly varying with the positive parameter $\gamma/p$. This concludes the proof.

$\square$

PROOF OF COROLLARY 12: Denote $r(n) := F^{-1}(1 - 1/n), \forall n > 1$ with $F^{-1}$ being the generalised inverse of the distribution function $F$. In view of Theorem 11, applying Proposition 0.8 (vii) of Resnick (1987), we have for $i \le k$

$$\lim_{n\to\infty} \frac{r(n)}{a_i(n)} \quad = \quad \left( \frac{\Gamma(\overline{\alpha}/p)\Gamma(\alpha_i + \gamma/p)}{2\Gamma(\alpha_i)\Gamma(\overline{\alpha} + \gamma/p)} \right)^{-1/\gamma} =: c_i.$$

Utilising the arguments presented in Theorem 11 we obtain for any $\boldsymbol{y} = (y_1, \dots, y_k)^\top > \boldsymbol{0}$

$$\lim_{n\to\infty} n\Big[1 - \boldsymbol{P}\{Y_1/a_1(n) \le y_1, \dots, Y_k/a_k(n) \le y_k\}\Big]$$
$$= \quad \lim_{n\to\infty} n\Big[1 - \boldsymbol{P}\Big\{ \frac{Y_1}{r(n)}\frac{r(n)}{a_1(n)} \le y_1, \dots, \frac{Y_k}{r(n)}\frac{r(n)}{a_k(n)} \le y_k\Big\}\Big]$$
$$= \quad \lim_{n\to\infty} n\Big[1 - \boldsymbol{P}\Big\{ c_1\frac{Y_1}{r(n)} \le y_1, \dots, c_k\frac{Y_k}{r(n)} \le y_k\Big\}\Big]$$
$$= \quad \gamma \int_0^\infty \boldsymbol{P}\{r\boldsymbol{c}A\boldsymbol{\mathcal{U}} \not\le \boldsymbol{y}\}r^{-\gamma-1}\, dr,$$

with $\boldsymbol{c} := (c_1, \dots, c_k)^\top$ and $\boldsymbol{\mathcal{U}} \sim \mathcal{GSD}(k, p, \boldsymbol{\alpha})$. This completes the proof. $\square$

## 8. APPENDIX

In this appendix several lemmas related to Dirichlet integrals, Gamma and Beta distributions are cited.

**Lemma 13.** *[Gupta and Song (1997), Lemma 1.1] Let $\boldsymbol{X}, \boldsymbol{Y}$ be two random vectors in $\mathbb{R}^k$ such that $\boldsymbol{X} \stackrel{d}{=} \boldsymbol{Y}$, and $f_i, 1 \le i \le d$, be measurable functions. We then have*

$$(45) \qquad (f_1(\boldsymbol{X}), \dots, f_d(\boldsymbol{X})) \stackrel{d}{=} (f_1(\boldsymbol{Y}), \dots, f_d(\boldsymbol{Y})).$$

**Lemma 14.** *[Gupta and Song (1997), Lemma 2.3] Let $f$ be a non-negative measurable function. For $\alpha_i > 0, i = 1, \dots, k$, we have:*

$$(46) \quad \int_{[0,\infty)^k} f\Big(\sum_{i=1}^k x_i\Big) \prod_{i=1}^k x_i^{\alpha_i-1}\, dx_1 \cdots dx_k \quad = \quad \frac{\prod_{i=1}^k \Gamma(\alpha_i)}{\Gamma(\overline{\alpha})} \int_0^\infty f(x)x^{\overline{\alpha}-1}\, dx,$$

*provided one of the integrals exist.*

The next lemma is a minor generalisation of Lemma 2.3 of Gupta and Song (1997).

**Lemma 15.** *Let $f$ be a non-negative measurable function. We then have for any $p_i > 0$ and $\alpha_i > 0, i = 1, \ldots, k$,*

$$(47) \quad \int_{[0,\infty)^k} f(\sum_{i=1}^k x_i^{p_i}) \prod_{i=1}^k x_i^{\alpha_i-1} \, dx_1 \cdots dx_k = \frac{\prod_{i=1}^k \Gamma(\alpha_i/p_i)}{\Gamma(\overline{\alpha}/p_i) \prod_{i=1}^k p_i} \int_0^\infty f(x) x^{\overline{\alpha}/p_i-1} \, dx$$

*and*

$$(48) \quad \int_{\mathbb{R}^k} f(\sum_{i=1}^k |x_i|^{p_i}) \prod_{i=1}^k |x|_i^{\alpha_i-1} \, dx_1 \cdots dx_k = \frac{2^k \prod_{i=1}^k \Gamma(\alpha_i/p_i)}{\Gamma(\overline{\alpha}/p_i) \prod_{i=1}^k p_i} \int_0^\infty f(x) x^{\overline{\alpha}/p_i-1} \, dx$$

*provided one of the integrals exist.*

*Proof.* Assume that the integral

$$I \quad := \quad \int_{[0,\infty)^k} f(\sum_{i=1}^k x_i^{p_i}) \prod_{i=1}^k x_i^{\alpha_i-1} \, dx_1 \cdots dx_k$$

is finite. Changing the variables $y_i := x_i^{p_i}, i \leq k$, and using Lemma 14 we obtain

$$\begin{aligned}
I &= \frac{1}{\prod_{i=1}^k p_i} \int_{[0,\infty)^k} f(\sum_{i=1}^k y_i) \prod_{i=1}^k y_i^{\alpha_i/p_i-1} \, dy_1 \cdots dy_k \\
&= \frac{\prod_{i=1}^k \Gamma(\alpha_i/p_i)}{\Gamma(\overline{\alpha}/p_i) \prod_{i=1}^k p_i} \int_0^\infty f(x) x^{\overline{\alpha}/p_i-1} \, dx.
\end{aligned}$$

Now the equation (48) follows easily.

$\square$

**Theorem 16.** *Let $Y$ be a random variable with the distribution function $H$ which has the upper endpoint $\omega \in (0, \infty]$ and $H(0) = 0$. Let $Z_{a,b}$ be a Beta distributed random variable with positive parameters $a, b$ being independent of $Y$, and $\tau > 0$ be a fixed constant.*
*i) If $H \in MDA(\Lambda)$ with a positive scaling function $w$ we have as $u \uparrow \omega$*

$$(49) \quad \boldsymbol{P}\{Y[1 - Z_{a,b}]^{1/\tau} > u\} \quad = \quad (1 + o(1))\frac{\Gamma(a+b)}{\Gamma(b)}\left(\frac{\tau}{uw(u)}\right)^a [1 - H(u)].$$

*ii) If $H \in MDA(\Phi_\alpha), \alpha > 0$, then $\omega = \infty$ and*

$$(50) \quad \boldsymbol{P}\{Y[1 - Z_{a,b}]^{1/\tau} > u\} \quad = \quad (1 + o(1))\frac{\Gamma(a+b)\Gamma(b+\alpha/\tau)}{\Gamma(b)\Gamma(a+b+\alpha/\tau)}[1 - H(u)]$$

*holds as $u \to \infty$*
*iii) If $H \in MDA(\Psi_\alpha), \alpha > 0$ and $\omega = 1$, we then have*

$$\begin{aligned}
&\boldsymbol{P}\{Y[1 - Z_{a,b}]^{1/\tau} > u\} \\
(51) \quad &= \quad (1 + o(1))\frac{\Gamma(\alpha+1)\Gamma(a+b)}{\Gamma(b)\Gamma(\alpha+a+1)}(\tau(1-u))^a[1 - H(u)], \quad u \uparrow 1.
\end{aligned}$$

*Proof.* The proof for the case $\tau = 2$ is given in Theorem 12.3.1, 12.3.2, 12.3.3 of Berman (1992). The general case $\tau > 0$ is shown in Theorem 6.2 of Hashorva (2006d). $\square$

**Lemma 17.** *Let $X, Y$ be two independent positive random variables with $Y^p \sim Gamma(a, \lambda), a, \lambda > 0, p > 0$. If $X$ is regularly varying with index $\gamma \geq 0$, we then have*

$$(52) \qquad \lim_{u \to \infty} \frac{\boldsymbol{P}\{XY > u\}}{\boldsymbol{P}\{Y > u\}} = \frac{\Gamma(a + \gamma/p)}{\lambda^{\gamma/p}\Gamma(a)} \in (0, \infty).$$

*Conversely, if the product $XY$ is regularly varying with index $\gamma \geq 0$, then $X$ is also regularly varying with index $\gamma$ and furthermore (52) is valid.*

*Proof.* The proof can be found in Lemma 6.1 of Hashorva (2006d) where the case $\gamma > 0$ is considered. We sketch it below. If $X$ is regularly varying with the positive index $\gamma \geq 0$, then (52) follows by Breiman's Lemma (see for some deep related results Denis and Zwart (2005)).

Suppose for simplicity that $p = 1, \lambda = 1$. For any $t > 0$ we may write by the independence of $X$ and $Y$

$$\boldsymbol{P}\{XY > t\} = \frac{t^a}{\Gamma(a)} \int_0^\infty \boldsymbol{P}\{XY > t | Y = tx\} \exp(-tx) x^{a-1} \, ds$$

$$= t^a \int_0^\infty \exp(-tv) \, dG(v),$$

where

$$G(s) := \frac{1}{\Gamma(a)} \int_0^s \boldsymbol{P}\{X > 1/x\} x^{a-1} \, dx, \quad s > 0.$$

The assumption $XY$ is regularly varying with index $\gamma \geq 0$ means

$$(53) \qquad \int_0^\infty \exp(-tv) \, dG(v) = t^{-a-\gamma} L(1/t), \quad t \to \infty,$$

with $L(x)$ such that $\lim_{t \to 0} L(Kt)/L(t) = 1, \forall K > 0$. In view of Karamata's Tauberian Theorem (Feller (1966), Resnick (1987)) (53) is equivalent with

$$G(t) = \frac{1}{\Gamma(a + \gamma + 1)} t^{a+\gamma} L(t), \quad t \to 0,$$

or equivalently

$$G(1/t) = \frac{1}{\Gamma(a + \gamma + 1)} t^{-a-\gamma} L(1/t), \quad t \to \infty.$$

Consequently as $t \to \infty$

$$\int_0^{1/t} \boldsymbol{P}\{X > 1/x\} x^{a-1} \, dx = \frac{\Gamma(a)}{\Gamma(a + \gamma + 1)} t^{-a-\gamma} L(1/t).$$

Since $\boldsymbol{P}\{X > x\} x^{-a-1}, x > 0$ decreases monotonically in $x$ for any $a > 0$ we get applying the Monotone Density Theorem (Resnick (1987))

$$\boldsymbol{P}\{X > t\} t^{-a-1} = \frac{(a + \gamma + 1)\Gamma(a)}{\Gamma(a + \gamma + 1)} t^{-a-\gamma-1} L(1/t), \quad t \to \infty,$$

thus the proof follows. $\qquad \square$

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